On the Review of Riemann and Riemann Stieljes Integrals

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

ABSTRACT

This work seeks an understanding of integration as a generalization of the summation process either in the Riemann or the Riemann Stieljes sense as the case may be.

First, on having an interval $[a, b]$ in $\mathbb{R}$, a partition is constructed with which the Riemann sums $R(f, p)$ is calculated and if such sums tends to a finite limit and the mesh $m(p)$ tends to zero then the function is intergrable and the Riemann integration of such function is defined as:

$$\sup_{p} L(f, p) = \inf_{p'} U(f, p')$$

for any partition $p \in \mathbb{R}$ and $p' \in [a, b] \in \mathbb{R}$ and $p'$ a refinement of $p$, where $U(f, p')$ and $L(f, p)$ have their usual meanings.

Again, with the interval $[a, b]$ and a partition on it such that the Riemann Stieljes sums of $f$ with respect to $\alpha, R(f, p, \alpha)$ are calculated and if such sums tends to a finite limit as the mesh $m(p)$ to zero then the function is Riemann Stieljes integrable and such integral is then defined as:

$$\sup_{p} L(f, p, \alpha) = \inf_{p'} U(f, p', \alpha)$$

for any partitions $p$ and $p'$ in $[a, b]$ and $p'$ a refinement of $p$, where $U(f, p', \alpha)$ and $L(f, p, \alpha)$ have their usual meanings.

Further, we explored the various properties of the Riemann and Riemann Stieljes integrals in form of theorems and lemma and in the last section we stated and proved some important and advanced results on the subject.

Keywords: Intervals; partition; infimum; supremum; lower sum; upper sum; integral.

1. THE RIEMANN INTEGRAL

1.1 Partitions and the Concept of Integral

We know that sometimes, integrals are usually understood to mean areas. However in this paper, we want it to be understood as a generalization of the summation process and hence the aim of this paper.

Definition 1.1 [1,2] Let $[a, b]$ be closed interval in $\mathbb{R}$. A finite, ordered set of points $\rho = \{x_0, x_1, x_2, \ldots, x_{k-1}, x_k\}$ such that

\[ x_0 = a \quad \text{and} \quad x_k = b \]

\[ x_i - x_{i-1} \neq 0 \quad \text{for} \quad i = 1, 2, \ldots, k \]
\[ a = x_0 \leq x_1 \leq x_2 \cdots x_{k-1} \leq x_k = b \]

is called a partition of \([a, b]\). Refer to Fig. 1.

If \( \rho \) is a partition of \([a, b]\), then we let \( l_j \) denote the interval \([x_{j-1}, x_j]\), \( j = 1, 2, \ldots, k \). The symbol \( \Delta_j \) denotes the length of \( l_j \). The mesh of \( \rho \), denoted by \( m(\rho) \) is defined to be \( \max \Delta_j \).

The points of partition need not be equally spaced, nor must they be distinct from each other.

**Definition 1.2** \([1,2]\) Let \([a, b]\) be an interval and let \( f \) be a function with domain \([a, b]\). If \( \rho = \{x_0, x_1, x_2, \ldots, x_{k-1}, x_k\} \) is a partition of \([a, b]\) and if for each \( j, s_j \) is an element of \( l_j \), then the corresponding Riemann sum is defined to be

\[
\mathcal{R}(f, \rho) = \sum_{j=1}^{k} f(s_j) \Delta_j.
\]

**Example 1.1** \([1,3]\) Let \( f(x) = x^2 - x \) and \([a, b] = [1,4] \). Define the partition \( \rho = \{1.3/2, 2, 7/3, 4\} \) of this interval. Then a Riemann sum for this \( f \) and \( \rho \) is

\[
\mathcal{R}(f, \rho) = (1^2 - 1) \cdot \frac{1}{2} + ((7/4)^2 - (7/4)) \cdot \frac{1}{3} + ((7/3)^2 - (7/3)) \cdot \frac{5}{12}
\]

\[
= \frac{0.0103}{8.64}
\]

Notice that we have complete latitude in choosing each point \( s_j \) from the corresponding interval \( l_j \), while at first confusing, we will find this freedom to be a powerful tool when proving results about the integral.

The first main step in theory of the Riemann integral is to determine a method for “calculating the limit of the Riemann sums” of a function as the mesh of partitions tends to zero. There are in fact several methods for doing this. We have chosen the simplest one.

**Definition 1.3** \([1,2]\) Let \([a, b]\) be an interval and \( f \) a function with domain \([a, b]\). We say that the Riemann sums of \( f \) tend to a limit \( \ell \) as \( m(\rho) \) tends to 0 if, for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that, if \( \rho \) is any partition of \([a, b]\) with \( m(\rho) < \delta \), then \( |\mathcal{R}(f, \rho) - \ell| < \epsilon \) for every choice of \( s_j \in l_j \).

It will turn out to be critical for the success of this definition that we require that every partition of mesh smaller than \( \delta \) satisfy the conclusion of the definition. The theory does not work effectively if for every \( \epsilon > 0 \) there is a \( \delta > 0 \) and some \( \rho \) of mesh less than \( \delta \) which satisfies the conclusion of the definition.

**Definition 1.4** \([1,1]\) A function \( f \) on a closed interval \([a, b]\) is said to be Riemann integrable on \([a, b]\) if the Riemann sums of \( \mathcal{R}(f, \rho) \) tend to a finite limit as \( m(\rho) \) tends to zero.

The value of the limit, when it exists, is called the Riemann integral of \( f \) over \([a, b]\) and is denoted by

\[
\int_{a}^{b} f(x) dx.
\]

**Remark 1.1** We mention now a useful fact that will be formalized in later section. Suppose that \( f \) is Riemann integrable on \([a, b]\) with the value of the integral being \( \ell \). Let \( \epsilon > 0 \). Then, as stated in the definition (with \( \epsilon/2 \) replacing \( \epsilon \)), there is a \( \delta > 0 \) such that if \( \rho \) is a partition of \([a, b]\) of mesh smaller than \( \delta \) then \( |\mathcal{R}(f, \rho) - \ell| < \epsilon/2 \). It follows that, if \( \rho \) and \( \rho' \) are partition of \([a, b]\) of mesh smaller than \( \delta \), then

\[
|\mathcal{R}(f, \rho) - \mathcal{R}(f, \rho')| \leq |\mathcal{R}(f, \rho) - \ell| + |\ell' - \mathcal{R}(f, \rho')| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

Note, however, that we may choose \( \rho' \) to equal the partition \( \rho \). Also we may for each \( j \) choose the points \( s_j \), where \( f \) is evaluated for the Riemann sum over \( \rho \), to be a point where \( f \) very nearly assumes its infimum on \( l_j \). It easily follows that when the mesh of \( \rho \) is less than \( \delta \) then

\[
\sum_{j} \left( \sup_{l_j} f - \inf_{l_j} f \right) \Delta_j \leq \epsilon \quad (1.1)
\]

This consequence of integrability will prove useful to us in some of the discussions in this and the next section. It is important to note that integrability implies (1.1) and the converse as well.

**Definition 1.5** \([1,3]\) If \( \rho, \rho' \) are partitions of \([a, b]\) then their common refinement is the union of all the points of \( \rho \) and \( \rho' \). (See Fig. 1).

We record now a technical lemma that will be used in several of the proofs that follow:
This equality enables us to rearrange (1.2) as

\[ \sum_{l_j \subseteq I_j} |f(s_j) - f(t_i)| \Delta_i \]

Now for each \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( \rho \) and \( \rho' \) are partitions of \([a, b]\) with \( m(\rho) < \delta \) and \( m(\rho') < \delta \) then their common refinement \( Q \) has the property

\[ |\mathcal{R}(f, \rho) - \mathcal{R}(f, Q)| < \varepsilon \]

and

\[ |\mathcal{R}(f, \rho') - \mathcal{R}(f, Q)| < \varepsilon \]

**Proof:** If \( f \) is Riemann integrable then the assertion of the lemma follows immediately from the definition of the integral.

For the converse note that (1.1) certainly implies that, if \( \varepsilon > 0 \) then there is a \( \sigma > 0 \) such that if \( \rho \) and \( \rho' \) are partitions of \([a, b]\) with \( m(\rho) < \delta \) and \( m(\rho') < \delta \), then

\[ |\mathcal{R}(f, \rho) - \mathcal{R}(f, \rho')| < \varepsilon. \]

(by just using the triangle inequality).

Now for each \( \varepsilon_j = 2^{-j}, j = 1, 2, \ldots \), we can choose a \( \delta_j > 0 \). Let \( S_j \) be the closure of the set

\[ \{ \mathcal{R}(f, \rho): m(\rho) < \delta_j \} \quad (1.2) \]

This equality enables us to rearrange (1.2) as

\[ \sum_{l_j \subseteq I_j} |f(s_j) - f(t_i)| \Delta_i \]

But each of the points \( t_i \) is in the interval \( I_j \), as is \( s_j \). So they differ by less than \( \delta \). Therefore, by (1.1), the last expression is less than

\[ \sum_{l_j \subseteq I_j} \frac{\varepsilon}{b-a} \Delta_i = \frac{\varepsilon}{b-a} \sum_{l_j \subseteq I_j} \Delta_i \]

Now we conclude the argument by writing

\[ |\mathcal{R}(f, \rho) - \mathcal{R}(f, Q)| \leq \sum_{j} \left| \sum_{l_j \subseteq I_j} f(s_j) \Delta_j - \sum_{l_j \subseteq I_j} f(t_i) \Delta_i \right| \]

\[ = \sum_{j} \left| \sum_{l_j \subseteq I_j} f(s_j) \Delta_j - \sum_{l_j \subseteq I_j} f(t_i) \Delta_i \right| < \sum_{j} \frac{\varepsilon}{b-a} \Delta_j \]

\[ = \frac{\varepsilon}{b-a} \cdot \sum_{j} \Delta_j = \frac{\varepsilon}{b-a} \cdot (b-a) = \varepsilon. \]

The estimate for \( |\mathcal{R}(f, \rho') - \mathcal{R}(f, Q)| \) is identical and we omit it. The result now follows from Lemma 1.1.

We conclude this section by noting an important fact about Riemann integrable functions. A Riemann integrable function on an interval \([a, b]\) must be bounded. If it were not, then one could choose \( s_j \) in the construction of \( \mathcal{R}(f, \rho) \) so that \( f s_j \) is arbitrarily large, and the Riemann sums would become arbitrarily large, hence cannot converge.
1.2 Properties of the Riemann Integral

We begin this subsection with a few elementary properties of the integral that reflect its linear nature.

Theorem 1.2 [2,4]

Let \([a, b]\) be a nonempty interval, let \(f\) and \(g\) be Riemann integrable function on the interval, and let \(\alpha\) be a real number. Then \(f \pm g\) and \(\alpha \cdot f\) are integrable and we have

\[
\int_a^b f(x) \pm g(x) \, dx = \int_a^b f(x) \, dx \pm \int_a^b g(x) \, dx;
\]

\[
\int_a^b \alpha \cdot f(x) \, dx = \alpha \cdot \int_a^b f(x) \, dx.
\]

Proof: For (a), let

\[
A = \int_a^b f(x) \, dx \quad \text{and} \quad B = \int_a^b g(x) \, dx;
\]

Let \(\varepsilon > 0\). Choose a \(\delta_1 > 0\) such that if \(\rho\) is a partition of \([a, b]\) with mesh less than \(\delta_1\) then

\[
|\mathcal{R}(f, \rho) - A| < \frac{\varepsilon}{2}
\]

Similarly choose a \(\delta_2 > 0\) such that if \(\rho\) is a partition of \([a, b]\) with mesh less than \(\delta_2\) then

\[
|\mathcal{R}(f, \rho) - B| < \frac{\varepsilon}{2}
\]

Let \(\delta = \min\{\delta_1, \delta_2\}\). If \(\rho'\) is any partition of \([a, b]\) with \(m(\rho') < \delta\) then

\[
|\mathcal{R}(f \pm g, \rho') - (A \pm B)| = |\mathcal{R}(f, \rho') \pm \mathcal{R}(g, \rho') - (A \pm B)|
\]

\[
|\mathcal{R}(f, \rho') - A| + |\mathcal{R}(g, \rho') - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

This means that the integral of \(f \pm g\) exists and equals \(A \pm B\) as we were required to prove.

The prove of (b) follows similar lines but is much easier.

Theorem 1.3 [5,4]

If \(c\) is a point of the interval \([a, b]\) and if \(f\) is Riemann integral on both \([a, c]\) and \([c, b]\) then \(f\) is integrable on \([a, b]\) and

\[
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

Proof: Let us write

\[
A = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx
\]

and

\[
B = \int_a^b f(x) \, dx.
\]

Now pick \(\varepsilon > 0\). There is a \(\delta_1 > 0\) such that if \(\rho\) is a partition of \([a, b]\) with mesh less than \(\delta_1\) then

\[
|\mathcal{R}(f, \rho) - A| < \frac{\varepsilon}{3}
\]

Similarly choose a \(\delta_2 > 0\) such that if \(\rho\) is a partition of \([a, b]\) with mesh less than \(\delta_2\) then

\[
|\mathcal{R}(f, \rho') - B| < \frac{\varepsilon}{3}
\]

Let \(M\) be an upper bound for \(|f|\) (recall, from the remark at the end of sub section 1, that a Riemann integrable function must be bounded): Set

\[
\delta = \min\{\delta_1, \delta_2, \varepsilon/(6M)\}.
\]

Now let \(\nu = \{v_1, \ldots, v_k\}\) be an partition \([a, b]\) with mesh less than \(\delta\).

There is a last point \(v_n\) which is in \([a, b]\) and a first point \(v_{n+1}\) in \([c, b]\). Observe that \(\rho = \{v_0, \ldots, v_n, c\}\) is a partition \([a, c]\) with mesh smaller than \(\delta_1\) and \(\rho' = \{c, v_n, \ldots, v_k\}\) is a partition \([c, b]\) with mesh smaller than \(\delta_2\). Let us rename the elements of \(\rho\) as \(\{p_0, \ldots, p_{n+1}\}\) and the elements \(\rho'\) as \(\{p'_0, \ldots, p'_{n+1}\}\). Notice that \(p_{n+1} = p'_0 = c\).

For each \(j\) let \(s_j\) be a point chosen in the interval

\[
I_j = [v_{j-1}, v_j]
\]

from the partition \(\nu\). Then we have

\[
|\mathcal{R}(f, \nu) - (A + B)| = \left| \left( \sum_{j=1}^{n} f(s_j) \Delta_j - A \right) + \left( \sum_{j=1}^{n} f(s_j) \Delta_j - B \right) \right|
\]

\[
= \left| \left( \sum_{j=1}^{n} f(s_j) \Delta_j + f(c) \cdot (c - v_n) - A \right) \right|
\]

\[
+ \left| f(c) \cdot (v_{n+1} - c) + \sum_{j=n+2}^{k} f(s_j) \Delta_j - B \right|
\]

\[
+ \left| (f(s_{n+1}) \cdot f(c)) \cdot (c - v_n) + (f(s_{n+1}) - f(c)) \cdot (v_{n+1} - c) \right|
\]

\[
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + 2M \cdot \delta \leq \varepsilon
\]
by the choice of \( \delta \).

This shows that \( f \) is integrable on the entire interval \([a, b]\) and the value of the integral is

\[
A + B = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx.
\]

**Remark 1.2** If we adopt the convention that

\[
\int_b^a f(x) \, dx = - \int_a^b f(x) \, dx
\]

(which is consistent with the way that the integral was defined in the first place). Then Theorem 1.3 is true even when \( c \) is not an element of \([a, b]\). For instance that \( c < a < b \), Then, by Theorem 1.3,

\[
\int_c^a f(x) \, dx + \int_a^b f(x) \, dx = \int_c^b f(x) \, dx
\]

But this may be rearranged to read

\[
\int_b^a f(x) \, dx = - \int_b^c f(x) \, dx + \int_c^b f(x) \, dx
\]

One of the basic tools of analysis is to perform estimates. Thus we require certain fundamental inequalities about integrals. These are recorded in the next theorem.

**Theorem 1.4** [5,4]

Let \( f \) and \( g \) be integrable function on a nonempty interval \([a, b]\). Then

\[
\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx;
\]

If \( f(x) \leq g(x) \) for all \( x \in [a, b] \) then \( \int_a^b f(x) \, dx \leq \int_a^b g(x) \, dx \).

**Proof:** If \( \rho \) is any partition of \([a, b]\) then

\[
|\mathcal{R}(f, \rho)| \leq \mathcal{R}(|f|, \rho).
\]

The first assertion follows.

Next, for part (ii),

\[
\mathcal{R}(f, \rho) \leq \mathcal{R}(g, \rho)
\]

This inequality implies the second assertion.

Another fundamental operation in the theory of the integral is “change of variable”.

We next turn to a careful formulation and proof of this operation. First we need a lemma:

**Lemma 1.2** [3,4]

If \( f \) is a Riemann integrable function on \([a, b]\) and if \( \emptyset \) is a continuous function on a compact interval that contains the range of \( f \) then \( \emptyset \circ f \) is Riemann integrable.

**Proof:** Let \( \varepsilon > 0 \). Since \( \emptyset \) is a continuous function on a compact set, it is uniformly continuous. Let \( \delta > 0 \) be selected such that (i) \( \delta < \varepsilon \) and (ii) if \( |x - y| < \delta \) then \( |\emptyset(x) - \emptyset(y)| < \varepsilon \).

Now the hypothesis that \( f \) is Riemann integrable implies that there exists a \( \delta > 0 \) such that if \( \rho \) and \( \rho' \), it holds that

\[
|\mathcal{R}(f, \rho) - \mathcal{R}(f, Q)| < \delta^2 \quad \text{and} \quad |\mathcal{R}(f, \rho) - \mathcal{R}(f, Q)| < \delta^2.
\]

Fix such that \( \rho, \rho' \) and \( Q \). Let \( j_\ell \) be the intervals of \( Q \) and \( i_\ell \) the intervals of \( \rho \). Each \( j_\ell \) is contained in some \( i_{\ell(\ell)} \). We write

\[
|\mathcal{R}(\emptyset \circ f, \rho) - \mathcal{R}(\emptyset \circ f, Q)|
\]

\[
= \left| \sum_{j} \emptyset \circ f(t_\ell) \Delta_{j} - \sum_{j} \emptyset \circ f(s_\ell) \Delta_{j} \right|
\]

\[
= \left| \sum_{j} \sum_{i \in j_\ell} \emptyset \circ f(t_\ell) \Delta_{j} - \sum_{j} \sum_{i \in j_\ell} \emptyset \circ f(s_\ell) \Delta_{j} \right|
\]

\[
= \left| \sum_{j} \sum_{i \in j_\ell, \ell \in G} \left[ \emptyset \circ f(t_\ell) - \emptyset \circ f(s_\ell) \right] \Delta_{j} \right|
\]

\[
\leq \left| \sum_{j} \sum_{i \in j_\ell, \ell \in G} \left[ \emptyset \circ f(t_\ell) - \emptyset \circ f(s_\ell) \right] \Delta_{j} \right|
\]

where we put \( \ell \) in \( G \) if \( j_\ell \subseteq i_{\ell(\ell)} \) and \( 0 \leq \left( \sup_{j_{\ell(\ell)}} f - \inf_{j_{\ell(\ell)}} f \right) < \delta \);
otherwise we put $\ell'$ into $B$. Notice that
\[
\sum_{\ell \in B} \delta \Delta \leq \left( \sup_{(a,b)} f - \inf_{(a,b)} f \right) \Delta \sum_{\ell \in B} \sum_{j \in I_{\ell}} \left( \sup_{(a,b)} f - \inf_{(a,b)} f \right) \Delta \epsilon
\]
\[
= \sum_{j=1}^{k} \left( \sup_{(a,b)} f - \inf_{(a,b)} f \right) \Delta \epsilon \leq \delta^2
\]
by the choice of $\tilde{\delta}$ (and Remark 1.1). Therefore
\[
\sum_{\ell \in B} \Delta \epsilon < \delta
\]
Let $M$ be an upper bound for $|\emptyset|$. Then
\[
\left| \sum_{j \in I_{\ell} : \ell \in B} (\emptyset \circ f(t_{j}) \circ f(s_{j})) \Delta \epsilon \right| \leq \sum_{j \in I_{\ell} : \ell \in B} (2 \cdot M) \Delta \epsilon
\]
\[
\leq 2 \cdot \delta \cdot M = 2M\epsilon
\]
Also
\[
\left| \sum_{j \in I_{\ell} : \ell \in B} (\emptyset \circ f(t_{j}) \circ f(s_{j})) \Delta \epsilon \right| \leq \sum_{j \in I_{\ell} : \ell \in B} \epsilon \Delta \epsilon
\]
since, for $\ell \in G$, we know that $|f(a) - f(b)| < \delta$
for any $a, b \in I_{\ell}(t)$. However, the last line does not exceed $(b - a) \cdot \epsilon$. Putting together our estimates, we find that
\[
\left| \mathcal{R}(\emptyset \circ f, \rho) - \mathcal{R}(\emptyset \circ f, Q) \right| < \epsilon \cdot (2M + (b - a))
\]
By symmetry, an analogous inequality holds for $\rho'$. By Lemma 1.1, this is what we needed to prove.

An easier result is that if $f$ is a Riemann integrable on an interval $[a, b]$ and if $\mu : [\alpha, \beta] \rightarrow [a, b]$ is continuous then $f \circ \mu$ is Riemann integrable.

**Corollary 1.1** [3,6]

If $f$ and $g$ are Riemann integrable on $[a, b]$, then so is the function $f \cdot g$.

**Proof:** By Theorem 1.2, $f + g$ is integrable. By the following lemma,
\[
(f + g)^2 = f^2 + 2f \cdot g + g^2
\]
is integrable. But the lemma also implies that $f^2$ and $g^2$ are integrable (here we use the function $\emptyset(x) = x^2$). It results, by subtraction, that $2 \cdot f \cdot g$ is integrable. Hence $f \cdot g$ is integrable.

**Theorem 1.5** [3,6]

Let $f$ be an integrable function on an interval $[a, b]$ of positive length. Let $\psi$ be a continuously differentiable function from another interval $[\alpha, \beta]$ of positive length into $[a, b]$, assume that $\psi$ is monotone increasing, one-to-one, and onto. Then
\[
\int_{a}^{b} f(x) \, dx = \int_{\alpha}^{\beta} f(\psi(x)) \cdot \psi'(x) \, dx
\]

**Proof:** Since $f$ is integrable, its absolute value is bounded by some number $M$. Fix $\epsilon > 0$. Since $\psi'$ is continuous on the compact interval $[\alpha, \beta]$, it is uniformly continuous. Hence we may choose $\delta > 0$ so small that if
\[
|s - t| < \delta \quad \text{then} \quad |\psi'(s) - \psi'(t)| < \epsilon / M \cdot (\beta - \alpha)
\]
If $\rho = \{\rho_{0}, ..., \rho_{k}\}$ is any partition of $[a, b]$ then there is an associated partition $\tilde{\rho} = \{\psi^{-1}(\rho_{0}), ..., \psi^{-1}(\rho_{k})\}$ of $[\alpha, \beta]$. For simplicity denote the point of $\tilde{\rho}$ by $\tilde{\rho}_{j}$. Let us choose the partition $\rho$ so fine that the mesh of $\tilde{\rho}$ is less than $\delta$. If $t_{j}$ are points of $I_{j} = [\rho_{j-1}, \rho_{j}]$, then there are corresponding points $s_{j} = \psi^{-1}(t_{j})$ of $I_{j} = [\tilde{\rho}_{j-1}, \tilde{\rho}_{j}]$. Then we have
\[
\sum_{j=1}^{k} f(t_{j}) \Delta = \sum_{j=1}^{k} f(t_{j})(\rho_{j} - \rho_{j-1})
\]
\[
= \sum_{j=1}^{k} \left( \psi(s_{j}) \right) (\psi(\tilde{\rho}_{j}) - \psi(\tilde{\rho}_{j-1}))
\]
\[
= \sum_{j=1}^{k} \left( \psi(s_{j}) \right) \psi'(u_{j})(\tilde{\rho}_{j} - \tilde{\rho}_{j-1})
\]
where we have used the Mean Value Theorem in the last line to find each $u_{j}$. Our problem at this point is that $f \circ \psi$ and $\psi'$ are evaluated at different points. So we must do some estimation to correct that problem.

The last display line equals
\[
\sum_{j=1}^{k} \left( \psi(s_{j}) \right) \psi'(s_{j})(\tilde{\rho}_{j} - \tilde{\rho}_{j-1}) + \sum_{j=1}^{k} \left( \psi(s_{j}) \right) \psi'(u_{j})
\]
\[
- \psi'(s_{j})(\tilde{\rho}_{j} - \tilde{\rho}_{j-1})
\]
The first sum is a Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ and the second sum is an error term. Since the points $u_j$ and $s_j$ are element of the same interval $I_j$ of length less than $\delta$, we conclude that $|\psi'(u_j) - \psi'(s_j)| < \varepsilon/(M \cdot |\beta - \alpha|)$. Thus the error term in absolute value does not exceed
\[
\sum_{j=1}^{k} \frac{M \cdot \varepsilon}{|\beta - \alpha|} \cdot (\hat{\beta}_j - \hat{\beta}_{j-1}) = \varepsilon
\]

This shows that every Riemann sum for $f$ on $[a, b]$ with sufficiently small mesh corresponds to a Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ plus an error term of size less than $\varepsilon$. A similar argument shows that every Riemann sum for $f(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ with sufficiently small mesh corresponds to a Riemann sum for $f$ on $[a, b]$ plus an error term of magnitude less than $\varepsilon$. The conclusion is that the integral of $f$ on $[a, b]$ (which exists by hypothesis) and the integral of $(\psi(x)) \cdot \psi'(x)$ on $[\alpha, \beta]$ (which exists by the corollary to lemma 1.2) agree.

We conclude this section with the very important

**Theorem 1.6** [The Fundamental Theorem of Calculus] [5,6]

Let $f$ be an integrable function on the interval $[a, b]$. For $x \in [a, b]$ we define

\[ F(x) = \int_{a}^{x} f(s) \, ds \]

If $f$ is continuous at $x \in (a, b)$ then

\[ F'(x) = f(x) \]

**Proof:** Fix $x \in (a, b)$. Let $\varepsilon > 0$. Choose, by the continuity of $f$ at $x$, a $\delta > 0$ such that $|s - x| < \delta$ implies $|f(s) - f(x)| < \varepsilon$. We may assume that $\delta < \min \{x - a, b - x\}$. If $|t - x| < \delta$ the

\[ \left| \frac{F(t) - F(x)}{t - x} - f(x) \right| = \left| \int_{a}^{t} f(s) \, ds - \int_{a}^{x} f(s) \, ds \right| \]

\[ = \left| \int_{x}^{t} f(s) \, ds - f(x) \right| \]

Notice that we rewrote $f(x)$ as the integral with respect to dummy variable $s$ over an interval of length $|t - x|$ divided by $(t - x)$. Assume for the moment that $t > x$. then the last line is dominated by

\[ \frac{\int_{x}^{t} f(s) \, ds - f(x)}{t - x} \leq \frac{\varepsilon}{t - x} \]

A similar estimate holds when $t < x$ (simply reverse the limits of integration). This shows that

\[ \lim_{t \to x} \frac{F(t) - F(x)}{t - x} = \varepsilon \]

exist and equals $f(x)$. Thus $F'(x)$ exists and equals $f(x)$.

**Corollary 1.2** [5,6]

If $f$ is a continuous function on $[a, b]$ and if $G$ is any continuously differentiable function on $[a, b]$ where their derivatives are of value $(a, b)$ then

\[ \int_{a}^{b} f(x) \, dx = G(b) - G(a) \]

**Proof:** Define $F$ as in the theorem. Since $F$ and $G$ have the same derivative on $(a, b)$, they differ by a constant. Then

\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a) = G(b) - G(a) \]

as desired.

**2. The Riemann Stieltjes' Integral**

For many purpose, such as integration by parts, it is natural to formulate the integral in more general context that we have considered in the first section. Our new information is called the Riemann-Stieltjes Integral and is described below.

Fix an interval $[a, b]$ on a monotonically increasing function $\alpha$ on $[a, b]$. If $\rho = \{\rho_0, \rho_1, \ldots, \rho_k\}$ is a partition of $[a, b]$, then let $\Delta_{\alpha j} = \alpha(\rho_j) - \alpha(\rho_{j-1})$. Let $f$ be a bounded function on $[a, b]$ and define the upper Riemann sum of $f$ with respect to $\alpha$ and the lower Riemann sum of $f$ with respect to $\alpha$ as follows:

\[ U(f, \rho, \alpha) = \sum_{j=1}^{k} M_j \Delta_{\alpha j} \]

and

\[ L(f, \rho, \alpha) = \sum_{j=1}^{k} m_j \Delta_{\alpha j} \]
Here the notation $M_j$ denotes the supremum of $f$ on the interval $I_j = [\rho_{j-1}, \rho_j]$ and $m_j$ denotes the infimum of $f$ on $I_j$.

In the special case $a(x) = x$ the Riemann sums discussed here have a form similar to the Riemann sums considered in the first two sections.

Moreover,

$$\mathcal{L}(f, \rho, \alpha) \leq R(f, \rho) \leq \mathcal{U}(f, \rho, \alpha).$$

We define

$$I^*(f) = \inf \mathcal{U}(f, \rho, \alpha)$$

and

$$I^*(f) = \sup \mathcal{L}(f, \rho, \alpha).$$

Here the supremum and infimum are taken with respect to all partitions of the interval $[a, b]$. These are, respectively, the upper and lower integrals of $f$ with respect to $\alpha$ on $[a, b]$.

By definition it is always true that, for any partition $\rho$

$$\mathcal{L}(f, \rho, \alpha) \leq I_\alpha(f) \leq I^*(f) \leq \mathcal{U}(f, \rho, \alpha).$$

It is natural to declare the integral to exist when the upper and lower integrals agree:

**Definition 2.1** [2,6] Let $\alpha$ be a monotone increasing function on the interval $[a, b]$ and let $f$ be bounded function on $[a, b]$. We say that the Riemann-Stieljtes integral of $f$ with respect to $\alpha$ exists if

$$I_\alpha(f) = I^*(f)$$

When the integral exists we denote it by

$$\int_a^b f \, d\alpha$$

Notice that the definition of Riemann-Stieljtes integral is different from the definition of Riemann integral that we used in the preceding section. It turns out that when $a(x) = x$ the two definitions are equivalent. In the present generality, it is easier to deal with upper and lower integral in order to determine the existence of integrals.

**Definition 2.2** [2,6] Let $\rho$ and $Q$ be partitions of $f$ the interval $[a, b]$. If each point of $\rho$ is also an element of $Q$ then we call $Q$ a refinement of $\rho$.

Notice that the refinement $Q$ is obtained by adding points to $\rho$. The mesh of $Q$ will be less than or equal to that of $\rho$. The following lemma enables us to deal effectively with our new language:

**Lemma 2.1** [2,6]

Let $\rho$ be a partition of interval $[a, b]$ and $f$ a function on $[a, b]$. Fix a monotone increasing function $\alpha$ on $[a, b]$. If $Q$ is a refinement of $\rho$ then

$$\mathcal{U}(f, Q, \alpha) \leq \mathcal{U}(f, \rho, \alpha)$$

and

$$\mathcal{L}(f, Q, \alpha) \leq \mathcal{L}(f, Q, \alpha).$$

**Proof:** Since $Q$ is a refinement of $\rho$ it holds that any interval $I_\ell$ arising from $Q$ is contained in some interval $I_{j(\ell)}$ arising from $\rho$. Let $M_\ell$ be the supremum of $f$ on $I_\ell$ and $M_{j(\ell)}$ the supremum of $f$ on the interval $I_{j(\ell)}$. Then $M_\ell \leq M_{j(\ell)}$. We conclude that

$$\mathcal{U}(f, Q, \alpha) = \sum_\ell M_\ell \Delta \alpha_\ell \leq \sum_\ell M_{j(\ell)} \Delta \alpha_\ell$$

we rewrite the right-hand side as

$$\sum_\ell M_{j(\ell)} \left( \sum_{i \in I_\ell} \Delta \alpha_i \right)$$

However, because $\alpha$ is monotone, the inner sum simply equals $\alpha(P_\ell) - \alpha(P_{\ell-1}) = \Delta \alpha_\ell$. Thus the last expression is to $\mathcal{U}(f, \rho, \alpha)$, as desired.

A similar argument applies to the lower sums

**Example 2.1** [1,6]

Let $[a, b] = [0, 10]$ and let $\alpha(x)$ be the greatest integer function. That is, $\alpha(x)$ is the greatest integer that does not exceed $x$. So for example, $\alpha(0.5) = 0$, $\alpha(2) = 2$ and $\alpha(-3/2) = -2$.

Certainly $\alpha$ is a monotone increasing function on $[0, 10]$. Let $f$ be any continuous function on. We shall determine whether
Let \( \rho \) be a partition of \([0,10]\). It is to our advantage to assume that the mesh of \( \rho \) is smaller than 1. Observe that \( \Delta \alpha_j \) equals the number of integers that lie in the interval \( I_j \) that is, either 0 or 1. Let \( I_{j_0}, I_{j_1}, \ldots, I_{j_{10}} \) be, in sequence, the intervals from the partition which do in fact contain each distinct integer (the first of these contains 0, the second contains 1, and so on up to 10). The

\[
\mathcal{U}(f, \rho, \alpha) = \sum_{r=0}^{10} M_{jr} \Delta \alpha_r = \sum_{r=1}^{10} M_{jr}
\]

and

\[
\mathcal{L}(f, \rho, \alpha) = \sum_{r=0}^{10} m_{jr} \Delta \alpha_r = \sum_{r=1}^{10} m_{jr}
\]

because any term in these sums corresponding to an interval not containing an integer must have \( \Delta \alpha_j = 0 \). Notice that \( \Delta \alpha_{j_0} = 0 \) since \( \alpha(0) = \alpha(P_1) = 0 \).

Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous on \([0,10]\), we may choose a \( \delta > 0 \) such that \( |s - t| < \delta \) implies that \( |f(s) - f(t)| < \varepsilon/20 \). If \( m(\rho) < \delta \) then it follows that \( |f(\ell) - M_{jr}| < \varepsilon/20 \) and \( |f(\ell) - m_{jr}| < \varepsilon/20 \) for \( \ell = 0, 1, \ldots, 10 \).

Therefore

\[
\mathcal{U}(f, \rho, \alpha) < \sum_{r=1}^{10} \left( f(\ell) + \frac{\varepsilon}{20} \right)
\]

and

\[
\mathcal{L}(f, \rho, \alpha) > \sum_{r=1}^{10} \left( f(\ell) - \frac{\varepsilon}{20} \right).
\]

Rearranging the first of these inequalities leads to

\[
\mathcal{U}(f, \rho, \alpha) < \left( \sum_{r=1}^{10} f(\ell) \right) + \frac{\varepsilon}{20}
\]

and

\[
\mathcal{L}(f, \rho, \alpha) > \left( \sum_{r=1}^{10} f(\ell) \right) - \frac{\varepsilon}{20}.
\]

Thus, since \( \mathcal{L}(f, \rho, \alpha) \) and \( \mathcal{U}(f, \rho, \alpha) \) are trapped between \( \mathcal{U} \) and \( \mathcal{L} \), we conclude that

\[
|\mathcal{L}(f, \rho, \alpha) - \mathcal{U}(f, \rho, \alpha)| < \varepsilon.
\]

We have seen that if the partition is line enough then the upper and lower integral of \( f \) with respect with to \( \alpha \) differ by at most \( \varepsilon \). It follows that \( \int_0^1 f \, d\alpha \) exists. Moreover,

\[
\left| \mathcal{L}(f, \rho, \alpha) - \sum_{r=1}^{10} f(\ell) \right| < \varepsilon
\]

and

\[
\left| \mathcal{U}(f, \rho, \alpha) - \sum_{r=1}^{10} f(\ell) \right| < \varepsilon.
\]

We conclude that

\[
\int_0^1 f \, d\alpha = \sum_{r=1}^{10} f(\ell)
\]

The example demonstrates that the language of the Riemann-Stieltjes integral allows us to think of the integral as a generalization of the summation process.

The next result, sometimes called Riemann’s lemma, is crucial for solving the existence of Riemann-Stieltjes integrals.

**Proposition 2.1 [1,6]**

Let \( \alpha \) be a monotone increasing function on \([a,b]\) and \( f \) a bounded function on the interval. The Riemann-Stieltjes integral of \( f \) with respect to \( \alpha \) exists if and only if, for every \( \varepsilon > 0 \), there is a partition \( \rho \) such that

\[
|\mathcal{U}(f, \rho, \alpha) - \mathcal{L}(f, \rho, \alpha)| < \varepsilon
\]

(2.1)

**Proof:** first assume that (2.1) holds. Fix \( \varepsilon > 0 \). Since \( \mathcal{L} \leq \mathcal{I} \leq \mathcal{U} \), inequality (2.1) implies that

\[
|\mathcal{I}(f) - \mathcal{I}^*(f)| < \varepsilon.
\]

But this means that \( \int_0^1 f \, d\alpha \) exists.

Conversely, assume that the integral exists. Fix \( \varepsilon > 0 \). Choose a partition \( \mathcal{Q}_1 \) such that
\[ |U(f, Q_1, \alpha) - I^*(f)| < \varepsilon/2. \]
Likewise choose a partition \( Q_2 \) such that
\[ |L(f, Q_2, \alpha) - I(f)| < \varepsilon/2. \]
Since \( I(f) = I^*(f) \) it follows that
\[ |U(f, Q_1, \alpha) - L(f, Q_2, \alpha)| < \varepsilon \] (2.2)

Let \( \rho \) be the common refinement of \( Q_1 \) and \( Q_2 \).
Then we have, again by lemma 2.1, that
\[ L(f, Q_2, \alpha) \leq L(f, \rho, \alpha) \leq \int_a^b f \, d\alpha \leq U(f, \rho, \alpha) \leq U(f, Q_1, \alpha). \]
But by (2.2), the expressions on the left and on the far right of these inequalities differ by less than \( \varepsilon \). Thus \( \rho \) satisfies the condition (2.1)

3. ADVANCED RESULTS

We now turn to establishing the existence of certain Riemann –Stieltjes integrals.

**Theorem 3.1 [5,6]**

Let \( f \) be continuous on \([a, b] \) and assume that \( \alpha \) is monotonically increasing. Then
\[ \int_a^b f \, d\alpha \]
exists.

**Proof:** we may assume that \( \alpha \) is nonconstant otherwise there is nothing to prove.

Pick \( \varepsilon > 0 \). By the uniform continuity of \( f \) we may choose a \( \delta > 0 \) such that if \( |s - t| < \delta \) then \( |f(s) - f(t)| < \varepsilon/(\alpha(b) - \alpha(a)) \). Let \( \rho \) be any partition of \([a, b] \) that has mesh smaller than \( \delta \). Then
\[ |U(f, \rho, \alpha) - L(f, \rho, \alpha)| = \sum_j |M_j \Delta_{a_j} - \sum_j m_j \Delta_{a_j}| \]
\[ = \sum_j |M_j - m_j| \Delta_{a_j} \]
\[ < \sum_j \frac{\varepsilon}{\alpha(b) - \alpha(a)} \Delta_{a_j} \]
\[ = \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_j \Delta_{a_j} \]
\[ = \varepsilon. \]

Here, of course, we have used the monotonicity of \( \alpha \) to observe that the last sum collapse to \( \alpha(b) - \alpha(a) \). By Riemann’s lemma, the proof is complete.

Notice how simple
\[ \int_a^b f \, d\alpha \]
Riemann’s lemma is to use. You may find it instructive to compare the proofs of this section with the rather difficult proofs in Section 2. What we are learning is that a good definition (and accompanying lemma(s)) can, in the end, make everything much simpler. Now we establish a companion result to the first one:

**Theorem 3.2**

If \( \alpha \) is a monotone increasing and continuous function on the interval \([a, b] \) and if \( f \) is monotonic on \([a, b] \) then \( \int_a^b f \, d\alpha \) exists.

**Proof:** We may assume that \( \alpha(b) > \alpha(a) \) and that \( f \) is monotone increasing. Let \( L = \alpha(b) - \alpha(a) \) and \( M = f(b) - f(a) \). Pick \( \varepsilon > 0 \). Choose \( k \) so that
\[ \frac{L \cdot M}{k} < \varepsilon. \]

Let \( P_0 = a \) and choose \( P_1 \) to be the first point to the right of \( P_0 \) such that \( \alpha(P_1) - \alpha(P_0) = L/k \) (this is possible, by the Intermediate Value Theorem, since \( \alpha \) is continuous). Continuing, choose \( P_{j+1} \) to be the first point to the right of \( P_j \) such that \( \alpha(P_{j+1}) - \alpha(P_j) = L/k \). This process will terminate after \( k \) steps and we will have \( P_k = b \). Then \( \rho = \{P_0, P_1, ..., P_k\} \) is a partition of \([a, b] \).

Next observe that, for each \( j \), the value \( M_j \) of sup \( f \) on \( I_j \) is \( f(P_j) \) since \( f \) is monotone increasing. Similarly the value \( m_j \) of \( \inf f \) on \( I_j \) is \( f(P_{j-1}) \), we find therefore that
\[ U(f, \rho, \alpha) - I(f, \rho, \alpha) = \sum_{j=1}^k M_j \Delta a_j - \sum_{j=1}^k m_j \Delta a_j \]
\[ = \sum_{j=1}^k \left( M_j - m_j \right) \frac{L}{k} \]
\[ = \frac{L}{k} \sum_{j=1}^k \left( f(P_j) - f(P_{j-1}) \right) \]
\[ = \frac{L}{k} \sum_{j=1}^k (f(x_j) - f(x_{j-1})) = \frac{L \cdot M}{k} < \varepsilon. \]

Therefore inequality (1.1) of Riemann’s lemma is satisfied and the integral exists.
One of the useful features of Riemann-Stieltjes integration is that it puts integration by parts into a very natural setting. We begin with a lemma:

**Lemma 3.1** [5,4]

Let $f$ be continuous on an interval $[a, b]$ and let $g$ be monotone increasing and continuous on that interval. If $G$ is an antiderivative for $g$ then

$$
\int_a^b f(x)g(x)\,dx = \int_a^b f\,dG.
$$

**Proof:** Apply the Mean Value Theorem to Riemann sums for the integral on the right.

**Theorem 3.3 [Integration by Parts]** [7,6]

Suppose that both $f$ and $g$ are continuous, monotone increasing functions on the interval $[a, b]$. Let $F$ be an antiderivative for $f$ on $[a, b]$ and $G$ an antiderivative for $g$ on $[a, b]$. Then we have

$$
\int_a^b FdG = [F(b) \cdot G(b) - F(a) \cdot G(a)] - \int_a^b GdF
$$

**Proof:** Notice that by preceding lemma, both integral exist. Set $P(x) = F(x) \cdot G(x)$. Then $P$ has a continuous derivative on the interval $[a, b]$. Thus the fundamental Theorem applies and we may write

$$
P(b) - P(a) = \int_a^b P'(x)dx = [F(b) \cdot G(b) - F(a) \cdot G(a)]
$$

Now writing out $P'$ explicitly, using Leibnitz’s Rule for the derivative of a product, we obtain

$$
\int_a^b F(x)g(x)\,dx = [F(b)G(b) - F(a)G(a)] - \int_a^b G(x)f(x)\,dx.
$$

But the lemma allows us to rewrite this equation as

$$
\int_a^b FdG = [F(b)G(b) - F(a)G(a)] = \int_a^b G(x)dF
$$

**Remark 3.1:** The integration by part formula can also be proved by applying summation by parts to Riemann sums for the integral

$$
\int_a^b f \,dG.
$$

We have already observed that the Riemann-Stieltjes integral

$$
\int_a^b f \,d\alpha
$$

is linear in $f$; that is

$$
\int_a^b (f + g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha
$$

and

$$
\int_a^b c \cdot f\,d\alpha = c \cdot \int_a^b f\,d\alpha
$$

when both $f$ and $g$ are Riemann-Stieltjes integrable with respect to $\alpha$ and for any constant $c$. We also would expect, from the very way that the integral is constructed, that it would be linear in the $\alpha$ entry. But we have not even defined the Riemann-Stieltjes integral for nonincreasing $\alpha$. And what of a function $\alpha$ that is the difference of two monotones. Is possible to identify which function $\alpha$ can be decomposed as sums or difference of monotonic functions? It turns out that there is a satisfactory answer to these questions, and we should like to discuss these matters briefly.

**Definition 3.1** If $\alpha$ is a monotonically decreasing function on $[a, b]$ and $f$ is a function on $[a, b]$ then we define

$$
\int_a^b f\,d\alpha = \int_a^b f\,d(-\alpha)
$$

when the right side exists.

The definition exploits the simple observation that if $\alpha$ is monotone decreasing then $-\alpha$ is monotone increasing; hence the preceding theory applies to the function $-\alpha$.

Next we have

**Definition 3.2** [8] Let $\alpha$ be a function on $[a, b]$ that can be expressed as $\int_a^b f\,d\alpha = \int_a^b f\,d\alpha_1 - \int_a^b f\,d\alpha_2$, provided that both integrals on the right exist.

Now, by the very way we have formulated our definitions, $\int_a^b f\,d\alpha$ is linear in both the $f$ entry and the $\alpha$ entry. But the definitions are not satisfactory unless we can identify those $\alpha$ that can actually occur in the last definition. This leads us to a new class of functions.

**Definition 3.3** [8] Let $f$ be a function on the interval $[a, b]$. For $x \in [a, b]$ we define
\[ Vf(x) = \sup \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})|, \]

where the supremum is taken over all partition \( \rho \) of the interval \([a, x]\).

If \( Vf \equiv Vf(b) < \infty \) then the function \( f \) is said to be of bounded variation on the interval \([a, b]\). In this circumstance the quantity \( Vf(b) \) is called the total variation of \( f \) on \([a, b]\).

A function of bounded variation has the property that its graph does not have unbounded total oscillation.

**Example 3.1** [5]

Define \( f(x) = \sin x \), with domain the interval \([0, 2\pi]\). Let us calculate \( Vf \). Let \( \rho \) be a partition of \([0, 2\pi]\). Since adding points to the partition only makes the sum \( \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| \) larger (by the triangle inequality), we may as well suppose that \( \rho = \{p_0, p_1, p_2, ..., p_k\} \) contains the point \( \pi/2, 3\pi/2 \). Say that \( p_{\ell_1} = \pi/2 \) and \( p_{\ell_2} = 3\pi/2 \). Then

\[
\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| = \sum_{j=1}^{\ell_1} |f(p_j) - f(p_{j-1})| + \sum_{j=\ell_1+1}^{\ell_2} |f(p_j) - f(p_{j-1})|
\]

However, \( f \) is monotone increasing on the interval \([0, 2\pi] = [0, p_{\ell_1}]\). Therefore the sum is just

\[
\sum_{j=1}^{\ell_1} f(p_j) - f(p_{j-1}) = f(p_{\ell_1}) - f(p_0) = f(\pi/2) - f(0) = 1
\]

Similarly, \( f \) is monotone on the intervals \([\pi/2, 3\pi/2 = [p_{\ell1}, p_{\ell2}]\) and \([\pi/2, 3\pi/2 = [p_{\ell2}, p_{\ellk}]\). Thus the second and third sums equal \( f(p_{\ell_1}) - f(p_{\ell_1}) = 2 \) and \( f(p_{\ell_2}) - f(p_{\ell_2}) = 1 \) respectively. It follows that

\[ Vf = Vf(b2\pi) = 1 + 2 + 1 = 4 \]

Of course \( Vf(x) \) for any \( x \in [0, 2\pi] \) can be computed by similar means.

In general, if \( f \) is a continuously differentiable function on an interval \([a, b]\) then

\[ Vf(x) = \int_a^x |f'(t)| \, dt \]

**Lemma 3.2** [2]

Let \( f \) be a function of bounded variation on the interval \([a, b]\). Then the function \( Vf - f \) is monotone increasing on the interval \([a, b]\).

**Proof:** Let \( s < t \) be element of \([a, b]\). Pick \( \varepsilon > 0 \). By the definition of \( Vf \) we may choose a partition \( \rho = \{p_0, p_1, p_2, ..., p_k\} \) of the interval \([a, s]\) such that

\[
Vf(s) - \varepsilon < \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| \tag{*}
\]

But then \( \bar{\rho} = \{p_0, p_1, p_2, ..., p_k\} \) is a partition of \([a, t]\) and we have that

\[
\sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| + |f(t) - f(s)| \leq Vf(t).
\]

Using (*), we may conclude that

\[
Vf(s) - \varepsilon + f(t) - f(s) < \sum_{j=1}^{k} |f(p_j) - f(p_{j-1})| + |f(t) - f(s)| \leq Vf(t).
\]

We conclude that

\[
Vf(s) - f(s) < Vf(t) - f(t) + \varepsilon.
\]

Since the inequality holds for every \( \varepsilon > 0 \), we see that the function \( Vf - f \) is monotone increasing.

Now we may combine the last two lemmas to obtain our main result.

**Proposition 3.1** [5]

If a function \( f \) is of bounded variation on \([a, b]\), then \( f = Vf - (Vf - f) \equiv f_1 - f_2 \). By the lemmas, both \( f_1 \) and \( f_2 \) are monotone increasing.

For the converse, assume that \( f = f_1 - f_2 \) with \( f_1, f_2 \) monotone increasing. Then it is easy to see that

\[
Vf(b) \leq |f_1(b) - f_1(a)| + |f_2(b) - f_2(a)|.
\]

Thus \( f \) is of bounded variation.
Now the main point of this discussion is the following theorem:

**Theorem 3.4 [7]**

If \( f \) is a continuous function on \([a, b]\) and if \( \alpha \) is of bounded variation on \([a, b]\) then the integral

\[
\int_{a}^{b} f \, d\alpha
\]

exist and is finite.

If \( g \) is of bounded variation on \([a, b]\) and if \( \beta \) is continuous function of bounded variation on \([a, b]\) then the integral

\[
\int_{a}^{b} g \, d\beta
\]

exist and is finite.

**COMPETING INTERESTS**

Author has declared that no competing interests exist.

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