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Deformation of Incompressible Hollow Cylindrical Structure Loaded by Azimuthal Shear

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Authors' contributions

This work was carried out in collaboration between both authors. The mathematical problem formulated and solved analytically by the authors. Author BEC performed analysis and simulation using software to determine variations of parameters and plot a graph, while author EBJ typed set and arrange the work in journal form. Both authors read and approved the final manuscript.

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ABSTRACT

The research is centred on isotropic, incompressible hollow cylindrical structure that is deforming under pure azimuthal shear. The mathematical formulation of angular displacement of the structure resulted into a non-linear second order ordinary differential equations. The resulting solutions of the boundary value problems were obtained through software analysis of the angular displacement. The effect of stress components and material parameters on the angular displacement of the structure were illustrated and compared.

Keywords: Pure azimuthal shear; isotropic material; angular displacement and elasticity.

1. INTRODUCTION

The study of incompressible materials (cylindrical or spherical) has been of significant interest in recent times due to its applications in mathematics, engineering fields and in solid mechanics. The applications include design and analysis of structures such as beams, plates and shell.

Rivlin [1] postulated basic foundation for elasticity, his work and that of other researchers have resulted in further studies on the deformation of incompressible and compressible materials. Hencky [2] analysed the properties of vulcanized rubber, general law of ideal elastic body and its relevance in sciences while Bekkedahl [3] studied forms of rubber as indicated by temperature – volume relationship where he correlated and interpreted data obtained and compared with other investigations on heat capacity, electrical properties and behaviour of rubber under stress.

Gent strain energy function incorporates material parameters such as hardening, softening and locking, the material parameter is introduced to account for the limited extensibility of rubber materials and induce rapid strain stiffening for large stretches. It also belongs to

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the class of generalized Neo Hookean material and depends only on the first strain invariant, Gent [4].

Jiang and Ogden [5] provided some new solutions for the axial shear of a circular cylindrical tube of compressible isotopic elastic material and discussed explicit solution for several forms of the strain energy function, analysing the plain strain characterized of finite torsion shear cylinder of a compressible elastic material.

Bechir et al. [6] studied the behaviour of isotropic and incompressible vulcanized natural rubbers and that of quasi-incompressible carbon black filled vulcanized natural rubbers considering both theoretically and experimentally obtained solutions by generalising the neo-Hookean model and derived an original form of the strain energy density function. The two-dimensional field of in-plane homogeneous displacements was determined using a cross-correlation technique. Zhu et al. [7] discussed the problem of the axisymmetric deformation of a thick-walled circular cylindrical elastic tube that is subject to pressure on its external boundaries and zero displacement on its ends is for an incompressible isotropic Neo-Hookean material which is highly nonlinear and can accommodate large strains and large displacements. The derived governing system of non-linear partial differential equation is solved with a computer based programming language.

Kassianidis et al. [8] studied the problem of azimuthal shear of a cylinder subject to finite deformation, a general form of strain energy function was used and a closed form solution was obtained for a reinforced Neo Hookean material which was used to determine the domain of strong ellipticity in terms of the relationship between the shear strain and the angle. Taking a different perspective in respect to the nature of the cylindrical material, the problem of azimuthal shear deformation of a transversely isotropic elastic cylindrical tube with small deformation was studied by Mohamed et al. [9], they considered four different versions based on the constraints of the fibres or tube materials and discovered that the conventional linear elasticity considerations associated with the perfectly flexible fibre assumptions cannot adequately account for effects of material anisotropy. The type of tube material looked into gave rise to four different versions which are susceptible to closed form solution when perfectly flexible, they discovered that the deformation patterns of fibres possessing bending and stiffness as well as stress distributions developed within the tube cross section fit the physical expectation much closer than perfectly flexible materials.

Merodio and Ogden [10] proposed a new example of the solution to the finite deformation boundary value problem for a residually stressed elastic body and combined extension, inflation and torsion of a circular cylindrical tube subject to radial and circumferential residual stress. The isochoric deformation consisted of axial extension, radial inflation and superimposed torsion which is formulated for a general elastic strain energy function. Two simple strain energy functions incorporating radial stress were used and the integral were evaluated to give close form expressions for pressure, axial load and torsional movement.

In addition to works done in elasticity on cylindrical materials, Darijani and Bahremen [11]applied polynomial hyperelastic models to obtain a closed form solution for analyses of rubbery solid circular cylinder while Robert et al. [12] with the use of neo-Hookean and the Mooney–Rivlin models found the strain energy function for isotropic incompressible solids demonstrating a linear relationship between shear stress and amount of shear, and between torque and amount of twist, when subject to large simple shear or torsion deformations.

The components of (the fourth order) elasticity tensor appearing in this theoretical formulation depends on the residual stress tensor components and the components of deformation gradient tensor. The approach acquired by is different from this more generalized approach since they used the linearized theory and a very specialized assumed form of the constitutive equation. Moniba [13] assumed the initial stress to be small in magnitude so that the terms are linear in the initial stress and made use of a different form of elasticity tensor in calculations. More recently, the reader is referred to the work done by. Basic equations for a residually-stressed elastic material are presented and also the discussion on the elasticity tensor for residually-stressed materials is carried out. The focus is on development of the constitutive law for an initially stressed material that has no intrinsic material symmetry, i.e. its response relative to the undeformed configuration is considered isotropic in the absence of initial stress.

The equation of the angular displacement resulted in second order nonlinear differential equations. The problem of solving the resultant second order nonlinear differential equations analytically to reduce errors is of great interest. The stress and displacement of an incompressible hollow cylindrical structure is determined after solving the resultant nonlinear second order differential equation.

1.1 Gaps and Objectives of Research

The aim of this work is to find analytical solution of the stresses and displacement of an incompressible hollow cylindrical tube undergoing pure azimuthal shear deformation.

The objectives of the work include derivation of field equations for incompressible solid deforming under torsion, deformation of Gent's material under azimuthal shear deformation, finding analytic solution to the derived nonlinear second order differential equation and using values for the mechanical parameter to derive analytic solutions and show the relationship between the parameter, displacement and stress.

2. FORMULATION OF THE PROBLEM

Let consider an elastic homogeneous isotropic cylindrical solid deforming under torsion. Also, let a point with cylindrical coordinates (R, Θ, Z) in the undeformed configuration map onto the point with cylindrical coordinates (r, θ, z) in the deformed configuration.

The pure azimuthal shear deformation of an incompressible cylindrical material is given by

$$r = \frac{R}{\lambda}\theta = \Theta + \tau(R) \quad \text{and} \quad z = Z \qquad (1)$$

where,

$\tau(R)$ is the torsion or twisting function to be determined,

λ and are positive constants which account for initial expansion and extension respectively, and $\gamma = \lambda$ for an incompressible material.

For the deformation (1), the deformation gradient tensor $\bar{F}$ is given by;

$$\bar{F} = \begin{pmatrix} \frac{dr}{dR} & \frac{1}{r}\frac{dr}{d\Theta} & \frac{dr}{dZ} \\ r\frac{d\theta}{dR} & \frac{r}{R}\frac{d\theta}{d\Theta} & r\frac{d\theta}{dZ} \\ \frac{dz}{dR} & \frac{1}{R}\frac{dz}{d\Theta} & \frac{dz}{dZ} \end{pmatrix} \qquad (2)$$

$$\bar{F} = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ r\,\tau_R & \frac{r}{R} & 0 \\ 0 & 0 & \end{pmatrix} \qquad (3)$$

The left Cauchy Green deformation tensor $\bar{B}$ is given by

$$\bar{B} = \bar{F}\bar{F}^T$$

Where the $\bar{F}^T$ is the transpose of $\bar{F}$

$$\bar{B} = \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ r\tau_R & \frac{r}{R} & 0 \\ 0 & 0 & \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda} & r\,\tau_R & 0 \\ 0 & \frac{r}{R} & 0 \\ 0 & 0 & \end{pmatrix}$$

$$\bar{B}^2 = \begin{pmatrix} (\frac{1}{\lambda})^4 + \left(\frac{r}{\lambda}\tau_R\right)^2 & \frac{r}{\lambda^3}\tau_r + \frac{r}{\lambda}\tau_r\left((r\tau_R)^2 + \left(\frac{r}{R}\right)^2\right) & 0 \\ \frac{r}{\lambda^3}\tau_R + \frac{r}{\lambda}\tau_r\left((r\tau_R)^2 + \left(\frac{r}{R}\right)^2\right) & \left(\frac{r}{\lambda}\tau_R\right)^2 + \left((r\tau_R)^2 + \left(\frac{r}{R}\right)^2\right)^2 & 0 \\ 0 & 0 & {}^4 \end{pmatrix} \qquad (4)$$

$$\bar{B}^{-1} = \left(\frac{r}{\lambda R}\right)^2 \begin{pmatrix} {}^2\left((r\tau_R)^2 + \left(\frac{r}{R}\right)^2\right) & {}^2\frac{r}{\lambda}\tau_R & 0 \\ {}^2\frac{r}{\lambda}\tau_R & {}^2(\frac{1}{\lambda})^2 & 0 \\ 0 & 0 & (\frac{1}{\lambda})^2\left((r\tau_R)^2 + \left(\frac{r}{R}\right)^2\right) - \left(\frac{r}{\lambda}\tau_R\right)^2 \end{pmatrix}$$

$$\overline{\mathrm{B}}^{-1} = \begin{pmatrix} \frac{{}^4\mathrm{r}\left((\mathrm{r}\tau_R)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)}{(\lambda\mathrm{R})^2} & \frac{{}^4\mathrm{r}^4\tau_R}{\lambda^4\mathrm{R}^2} & 0 \\ \frac{{}^4\mathrm{r}^4\tau_R}{\lambda^4\mathrm{R}^2} & \frac{{}^4\mathrm{r}^4}{\lambda^4\mathrm{R}^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^4}(\frac{\mathrm{r}}{\mathrm{R}})^2\left((\mathrm{r}\tau_R)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)-\frac{1}{\lambda^4}\left(\frac{\mathrm{r}}{\mathrm{R}}\tau_R\right)^2 \end{pmatrix} \quad (5)$$

The principal strain invariants I_1, I_2 and I_3 are given by

$$\mathrm{I}_1 = \text{trace } \overline{\mathrm{B}} = (\tfrac{1}{\lambda})^2 + {}^2 + (\mathrm{r}\tau_R)^2 + \left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2 \quad (6)$$

$$\mathrm{I}_2=½[(tr\overline{\mathrm{B}})^2 - tr\overline{\mathrm{B}}^2=½[\left((\tfrac{1}{\lambda})^2 + {}^2 + (\mathrm{r}\tau_R)^2 + \left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)^2 - \left((\tfrac{1}{\lambda})^4 + 2\left(\frac{\mathrm{r}}{\lambda}\tau_R\right)^2 + (\mathrm{r}\tau_R)^4 + 2(\mathrm{r}\tau_R)^2\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2 + \right.$$
$\mathrm{rR4}+$ ▯4

$$\mathrm{I}_3 = \det \overline{\mathrm{B}} = \left(\frac{\lambda\mathrm{R}}{\mathrm{r}}\right)^2 \quad (7)$$

The Cauchy stress tensor σ for an isotropic incompressible hyperelastic solid is given by

$$\sigma = -\mathrm{PI} + 2\mathrm{W}_1\overline{\mathrm{B}} - 2\mathrm{W}_2\overline{\mathrm{B}}^{-1} \quad (8)$$

where I is the unit tensor , W is the strain energy density function and $\mathrm{W_i} = \frac{\mathrm{dW}}{\mathrm{dI_i}}$which depends on the invariants and $-\mathrm{P}$ is the hydrostatic pressure .

In component form, the Cauchy stress tensor is

$$\sigma = -\mathrm{P}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} + 2\mathrm{W}_1\begin{pmatrix}(\frac{1}{\lambda})^2 & \frac{\mathrm{r}}{\lambda}\tau_r & 0\\ \frac{\mathrm{r}}{\lambda}\tau_r & (\mathrm{r}\tau_r)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2 & 0 \\ 0 & 0 & {}^2\end{pmatrix} -$$
$$2\mathrm{W}_2\begin{pmatrix} \frac{{}^4\mathrm{r}\left((\mathrm{r}\tau_r)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)}{(\lambda\mathrm{R})^2} & \frac{{}^4\mathrm{r}^4\tau_r}{\lambda^4\mathrm{R}^2} & 0 \\ \frac{{}^4\mathrm{r}^4\tau_r}{\lambda^4\mathrm{R}^2} & \frac{{}^4\mathrm{r}^4}{\lambda^4\mathrm{R}^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^4}(\frac{\mathrm{r}}{\mathrm{R}})^2\left((\mathrm{r}\tau_r)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)-\frac{1}{\lambda^4}\left(\frac{\mathrm{r}}{\mathrm{R}}\tau_r\right)^2 \end{pmatrix}$$

$$\sigma = \begin{pmatrix}-\mathrm{P}&0&0\\0&-\mathrm{P}&0\\0&0&-\mathrm{P}\end{pmatrix} + 2\mathrm{W}_1\begin{pmatrix}(\frac{1}{\lambda})^2 & \frac{\mathrm{r}}{\lambda}\tau_R & 0\\ \frac{\mathrm{r}}{\lambda}\tau_R & (\mathrm{r}\tau_R)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2 & 0 \\ 0 & 0 & {}^2\end{pmatrix} -$$
$$2\mathrm{W}_2\begin{pmatrix} \frac{{}^4\mathrm{r}\left((\mathrm{r}\tau_R)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)}{(\lambda\mathrm{R})^2} & \frac{{}^4\mathrm{r}^4\tau_R}{\lambda^4\mathrm{R}^2} & 0 \\ \frac{{}^4\mathrm{r}^4\tau_R}{\lambda^4\mathrm{R}^2} & \frac{{}^4\mathrm{r}^4}{\lambda^4\mathrm{R}^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^4}(\frac{\mathrm{r}}{\mathrm{R}})^2\left((\mathrm{r}\tau_R)^2+\left(\frac{\mathrm{r}}{\mathrm{R}}\right)^2\right)-\frac{1}{\lambda^4}\left(\frac{\mathrm{r}}{\mathrm{R}}\tau_R\right)^2 \end{pmatrix} \quad (9)$$

3. FINITE EQUATION FOR AN INCOMPRESSIBLE HOLLOW CYLINDER UNDERGOING PURE AZIMUTHUAL SHEAR DEFORMATION

Considering an inextensible incompressible material where $\gamma=\lambda = 1$

Let the relationship under azimuthal shear deformation be described by

$$r = R \quad \theta = \Theta + g(R) \quad z = Z \tag{10}$$

Where we replace τ with g for convenience

From (2) the deformation gradient is given by

$$\overline{F}=\begin{pmatrix} \frac{dr}{dR} & \frac{1}{r}\frac{dr}{d\Theta} & \frac{dr}{dZ} \\ r\frac{d\theta}{dR} & \frac{r}{R}\frac{d\theta}{d\Theta} & r\frac{d\theta}{dZ} \\ \frac{dz}{dR} & \frac{1}{R}\frac{dz}{d\Theta} & \frac{dz}{dZ} \end{pmatrix} \tag{11}$$

Using (10) in (11), we have

$$\overline{F}=\begin{pmatrix} 1 & 0 & 0 \\ rg_R & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since r= R from (10), we have

$$\overline{F}=\begin{pmatrix} 1 & 0 & 0 \\ rg_R & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{12}$$

The transpose of F is

$$\overline{F}^T=\begin{pmatrix} 1 & rg_R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13}$$

$$F\overline{F}^T=\begin{pmatrix} 1 & 0 & 0 \\ rg_R & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & rg_R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & rg_R & 0 \\ rg_R & (rg_R)^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{14}$$

The left Cauchy Green deformation tensor is

$$\overline{B}=F\overline{F}^T=\begin{pmatrix} 1 & 0 & 0 \\ rg_R & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & rg_R & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & rg_R & 0 \\ rg_R & (rg_R)^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{15}$$

$$\overline{B}^2=\begin{pmatrix} (rg_R)^2+1 & (rg_R)^3+2rg_R & 0 \\ (rg_R)^3+2rg_R & (rg_R)^4+3(rg_R)^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{16}$$

$$\overline{B}^{-1}=\begin{pmatrix} (rg_R)^2+1 & rg_R & 0 \\ rg_R & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

Principal strain invariant

$$I_1 = \text{trace } \overline{B} = 3 + (rg_R)^2 \tag{18}$$

$$I_2= ½ [(tr\overline{B})^2 - tr\overline{B}^2] =½[(3 + (rg_R)^2)^2 - ((rg_R)^4 + 4(rg_R)^2 + 3)]$$
$$=½[(9 + 6(rg_R)^2 + (rg_R)^4 - ((rg_R)^4 + 4(rg_R)^2 + 3)]$$
$$= 3 + (rg_R)^2 \tag{19}$$

$$I_3 = \det \overline{B} = 1 \tag{20}$$

The third strain invariant shows that the material is incompressible.

The Cauchy stress tensor is given as

$$\sigma = -PI + 2W_1\overline{B} - 2W_2\overline{B}^{-1} \qquad \text{from (8)}$$

$$W_i= \frac{dW}{dI_i} W_1 = \frac{dW}{dI_1} W_2= \frac{dW}{dI_2}$$

$$I_1= I_2 \quad \sigma = PI2W_1(\overline{B} - \overline{B}^{-1})$$

$$\overline{B} - \overline{B}^{-1} = \begin{pmatrix} (rg_R)^2 & 0 & 0 \\ 0 & (rg_R)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} 2W_1(rg_R)^2 & 0 & 0 \\ 0 & 2W_1(rg_R)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

Since the invariants $I_1= I_2$, this suggest the use of a strain energy function dependent on the first strain invariant only. Hence we shall be considering the Gent's strain energy function.

The Gent's strain energy function W for nonlinear elastic behaviour of rubber like material is given by

$$W= -\frac{\mu J_m}{2} In\left(1 - \frac{I_1-3}{J_m}\right) \tag{22}$$

where μ is shear modulus

J_m is the material parameter for mechanical properties such as hardening and softening of the material

I_1 is the first principal strain invariant

$I_1= I_2$ implies that $W_1 = W_2$, so (8) becomes

$$\sigma = -PI + 2W_1\overline{B}$$

$$\sigma = -PI + \frac{\mu J_m}{J_m - (I_1 - 3)}\overline{B} \tag{23}$$

$$\sigma = -PI + \frac{\mu J_m}{J_m-(I_1-3)}\begin{pmatrix} 1 & rg_R & 0 \\ rg_R & (rg_R)^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (24)$$

$$\sigma = \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{zr} & \sigma_{z\theta} & \sigma_{zz} \end{pmatrix}$$

The physical components of the stress tensor becomes

$$\sigma_{rr} = -P + \frac{\mu J_m}{J_m-(3+(rg_R)^2-3)}$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m((rg_R)^2+1)}{J_m-(3+(rg_R)^2-3)}$$

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu J_m(rg_R)}{J_m-(3+(rg_R)^2-3)}$$

$$\sigma_{zr} = \sigma_{rz} = -P + \frac{\mu J_m}{J_m-(3+(rg_R)^2-3)}(0) = 0$$

$$\sigma_{\theta z} = \sigma_{z\theta} = -P + \frac{\mu J_m}{J_m-(3+(rg_R)^2-3)}(0) = 0$$

$$\sigma_{zz} = -P + \frac{\mu J_m}{J_m-(3+(rg_R)^2-3)}$$

The resulting equations are

$$\sigma_{rr} = -P + \frac{\mu J_m}{J_m-(rg_R)^2} = \sigma_{zz} \quad (25)$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m((rg_R)^2+1)}{J_m-(rg_R)^2} \quad (26)$$

$$\sigma_{\theta r} = \sigma_{r\theta} = P + \frac{\mu J_m(rg_R)}{J_m-(rg_R)^2} \quad (27)$$

$$\sigma_{zr} = \sigma_{rz} = -P + \frac{\mu J_m}{J_m-((rg_R)^2}(0) = 0 \quad (28)$$

$$\sigma_{\theta z} = \sigma_{z\theta} = -P + \frac{\mu J_m}{J_m-((rg_R)^2}(0) = 0 \quad (29)$$

The equilibrium equations in cylindrical polar coordinates is given by

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\frac{d\sigma_{r\theta}}{d\theta} + \frac{d\sigma_{rz}}{dz} + \frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}) + \rho b_r = \rho a_r$$

$$\frac{d\sigma_{\theta r}}{dr} + \frac{1}{r}\frac{d\sigma_{\theta\theta}}{d\theta} + \frac{d\sigma_{\theta z}}{dz} + \frac{2}{r}\sigma_{r\theta} + \rho b_\theta = \rho a_\theta$$

$$\frac{d\sigma_{zr}}{dr} + \frac{1}{r}\frac{d\sigma_{z\theta}}{d\theta} + \frac{d\sigma_{zz}}{dz} + \frac{1}{r}\sigma_{zr} + \rho b_z = \rho a_z$$

where

b_r, b_θ and b_z are the components of body force

a_r, a_θ and a_z are the componenets of acceleration

The equilibrium equation in the absence of any body force is reduced to

$$\frac{d\sigma_{rr}}{dr} + \frac{1}{r}\frac{d\sigma_{r\theta}}{d\theta} + \frac{d\sigma_{rz}}{dz} + \frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}) = 0 \quad (30)$$

$$\frac{d\sigma_{\theta r}}{dr} + \frac{1}{r}\frac{d\sigma_{\theta\theta}}{d\theta} + \frac{d\sigma_{\theta z}}{dz} + \frac{2}{r}\sigma_{r\theta} = 0$$

$$\frac{d\sigma_{zr}}{dr} + \frac{1}{r}\frac{d\sigma_{z\theta}}{d\theta} + \frac{d\sigma_{zz}}{dz} + \frac{1}{r}\sigma_{zr} = 0 \quad (32)$$

Differentiating (32) with respect with r,R and Using (25) - (29) in (30) - (32)

$$\frac{d\sigma_{rr}}{dr} = \frac{1}{r}(\sigma_{\theta\theta}-\sigma_{rr})$$

$$\frac{d}{dr}\left(-P + \frac{\mu J_m}{J_m-(rg_r)^2}\right) = \frac{1}{r}\left\{\left(-P + \frac{\mu J_m}{J_m-(rg_r)^2}((rg_r)^2+1)\right) - \left(-P + \frac{\mu J_m}{J_m-(rg_r)^2}\right)\right\}$$

$$\frac{d}{dr}\left(-P + \frac{\mu J_m}{J_m-(rg_r)^2}\right) = \frac{1}{r}\left(\frac{\mu J_m(rg_r)^2}{J_m-(rg_r)^2}\right) \quad (33)$$

$$\frac{d\sigma_{\theta r}}{dr} + \frac{2}{r}\sigma_{r\theta} = 0$$

$$\frac{d}{dr}\left(-P + \frac{\mu J_m(rg_r)}{J_m-(rg_r)^2}\right) = -\frac{2}{r}\left(-P + \frac{\mu J_m(rg_r)}{J_m-(rg_r)^2}\right) \quad (34)$$

Using (33)

$$\frac{d}{dR}\left(-P + \frac{\mu J_m}{J_m-(Rg_R)^2}\right) = \frac{1}{R}\left(\frac{\mu J_m(Rg_R)^2}{J_m-(Rg_R)^2}\right)$$

Where r = R from (10)

$$\frac{\mu J_m(2Rg_R^2+2R^2g_{RR}g_R)}{(J_m-(Rg_R)^2)^2}=\frac{\mu J_m Rg_R^2}{J_m-(Rg_R)^2}$$

$$\frac{\mu J_m(2Rg_R^2+2R^2g_{RR}g_R)}{J_m-(Rg_R)^2}=\mu J_m Rg_R^2$$

$$\frac{\mu J_m Rg_R(2g_R+2Rg_{RR})}{\mu J_m Rg_R^2}=J_m-(Rg_R)^2$$

$$\frac{2(g_R+Rg_{RR})}{g_R}=J_m-(Rg_R)^2$$

Therefore

$$g_{RR}=\frac{J_m}{2R}g_R-\frac{R}{2}g_R^3-\frac{1}{R}g_R$$

$$g_{RR}=\left(\frac{J_m}{2R}-\frac{1}{R}\right)g_R\quad-\frac{R}{2}g_R^3$$

$$g_{RR}=\frac{1}{R}\left(\frac{J_m}{2}-1\right)g_R\quad-\frac{R}{2}g_R^3$$

where

$$\beta=\left(\frac{J_m}{2}-1\right) \tag{35}$$

$$2Rg_{RR}=2\beta g_R-R^2g_R^3$$

$$2Rg_{RR}-2\beta g_R+R^2g_R^3=0 \tag{36}$$

4. ANALYSIS AND RESULTS

Let consider cylindrical structure under azimuthal shear deformation as a result of the shearing force on a plane perpendicular to the axis of a hollow cylinder such that the internal and external surfaces are stress free.

This leads to the boundary value function;

$$\begin{cases}2Rg_{RR}-2\beta g_R+R^2g_R^3=0\\ \text{g(a)}=0\quad \text{g(b)}=\Psi\end{cases} \tag{37}$$

Where a and b represent the internal and external radii before deformation.

Solving (37) with the boundary conditions in (37) and assuming;

$$E=g_R;\ \frac{dE}{dR}=g_{RR} \tag{38}$$

Then (37a) becomes

$$2R\frac{dE}{dR}-2\beta E+R^2E^3=0,$$

then

$$-2E^{-3}\frac{dE}{dR}+\frac{2\beta E^{-2}}{R}=R \tag{39}$$

By setting

Let $h=E^{-2}$

$\frac{dh}{dE}=-2E^{-3}\frac{dE}{dR}$ Substituting (39) in (38) we have

$$\frac{dh}{dE}+\frac{2\beta}{R}h=R \tag{40}$$

From equ (41), integrating factor (IF) is given as;

$$\text{IF}=e^{\int\frac{2\beta}{R}dR}=R^{2\beta}$$

Clearly (40) is equivalent to

$$h\ \text{IF}=\int R\ IF dR+C_1$$

$$h(R^{2\beta})=\int R.R^{2\beta}dR+C_1$$

$$h(R^{2\beta})=\frac{(R^{2+2\beta})}{(2+2\beta)}+C_1$$

$$h=\frac{(R^{2+2\beta})}{R^{2\beta}(2+2\beta)}+\frac{C_1}{R^{2\beta}}$$

Recall that from (4.4), $h=E^{-2}$

$$E^{-2}=\frac{(R^{2\beta+2})}{R^{2\beta}(2+2\beta)}+\frac{C_1}{R^{2\beta}}$$

$$E=\frac{\sqrt{(2\beta+2)R^{2\beta}}}{\sqrt{(R^{2\beta+2})+C_1(2\beta+2)}}$$

Also, from equ (38), $E=g_R=\frac{dg}{dR}$

Hence,

$$\frac{dg}{dR}=\frac{\sqrt{(2\beta+2)R^{2\beta}}}{\sqrt{(R^{2\beta+2})+C_1(2\beta+2)}}$$
$$=\frac{\sqrt{2}\sqrt{(\beta+1)R^{2\beta}}}{\sqrt{(R^{2\beta+2})+C_1(2\beta+2)}}$$

Therefore

$$g = \int \frac{\sqrt{(2\beta+2)R^{2\beta}}}{\sqrt{(R^{2+2\beta})+C_1(2\beta+2)}} dR + C_2$$

Using the Wolfram Alpha software, we obtain:

$$g(R) = \frac{\sqrt{C_1(2\beta+2)}R^{\beta}\sqrt{\frac{2R^{(2+2\beta)}}{C_1(2+2\beta)}+2}\ \sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}}}{\sqrt{(\beta+1)R^{2\beta}}\sqrt{C_1(2\beta+2)+R^{2\beta+2}}} + C_2 \quad (41)$$

$$g(R) = \frac{\sqrt{C_1(2\beta+2)}R^{\beta}\sqrt{\frac{2R^{(2+2\beta)}+2C_1(2+2\beta)}{C_1(2+2\beta)}}\sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}}}{\sqrt{(\beta+1)R^{2\beta}}\sqrt{C_1(2\beta+2)+R^{2\beta+2}}} + C_2$$

$$g(R) = \frac{\sqrt{C_1(2\beta+2)}R^{\beta}\frac{\sqrt{2R^{(2+2\beta)}+2C_1(2+2\beta)}}{\sqrt{C_1(2\beta+2)}}\sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}}}{R^{\beta}\sqrt{\beta+1}\sqrt{C_1(2\beta+2)+R^{2\beta+2}}} + C_2$$

$$g(R) = \frac{R^{\beta}\sqrt{2R^{(2+2\beta)}+2C_1(2+2\beta)}\sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}}}{R^{\beta}\sqrt{\beta+1}\sqrt{C_1(2\beta+2)+R^{2\beta+2}}} + C_2$$

$$g(R) = \frac{\sqrt{2}\sqrt{R^{(2+2\beta)}+C_1(2+2\beta)}\sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}}}{\sqrt{\beta+1}\sqrt{C_1(2\beta+2)+R^{2\beta+2}}} + C_2$$

Simplifying, we have

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{R^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + C_2 \quad (42)$$

where $C_1 and C_2$ are constants of integration.

Using the boundary conditions (37b) we obtain the following equations for the determination of the integration constants.
(44) and (45) are derived using the boundary conditions on (37)

$$g\,(a) = \frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + C_2 = 0 \quad (43)$$

$$g(b) = \frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + C_2 = \Psi \quad (44)$$

Equations (44) and (45) are solved for C_1 and C_2. Thus

$$C_2 = -\frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}} \quad (45)$$

Using (45) and (46)

$$\Psi = \frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} - \frac{\sqrt{2}}{\sqrt{\beta+1}}\sinh^{-1}\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}$$

$$\Psi = \frac{\sqrt{2}}{\sqrt{\beta+1}}\left(\sinh^{-1}\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} - \sinh^{-1}\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)$$

$$\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}} = \left(\sinh^{-1}\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} - \sinh^{-1}\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)$$

$$\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}} = \ln\left\{\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2+1}\right\} - \ln\left\{\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2+1}\right\}$$

$$\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}} = \ln\left(\frac{\left(\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2+1}\right)}{\left(\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2+1}\right)}\right)$$

$$e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} = \frac{\left(\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{b^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2 + 1}\right)}{\left(\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}} + \sqrt{\left(\frac{a^{\beta+1}}{\sqrt{C_1(2+2\beta)}}\right)^2 + 1}\right)}$$

$$e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} = \frac{\frac{b^{\beta+1} + \sqrt{b^{2\beta+2} + C_1(2+2\beta)}}{\sqrt{C_1(2+2\beta)}}}{\frac{a^{\beta+1} + \sqrt{a^{2\beta+2} + C_1(2+2\beta)}}{\sqrt{C_1(2+2\beta)}}}$$

$$e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} = \frac{b^{\beta+1} + \sqrt{b^{2\beta+2} + C_1(2+2\beta)}}{a^{\beta+1} + \sqrt{a^{2\beta+2} + C_1(2+2\beta)}} \qquad (46)$$

Solving equ(47) with the Wolfram Alpha software, we obtain:

Where A $= a^{2\beta+2}\left(e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}}\right) + b^{2\beta+2}\left(e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}}\right) -$ *aβ+1 bβ+1eΨβ+12−aβ+1 bβ+1*

$$C_1 = \frac{2e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}{(\beta+1)(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)^2} \qquad (47)$$

Using (46) in (48), we obtain

$$C_2 = -\frac{\sqrt{2}}{\sqrt{\beta+1}} \sinh^{-1} \frac{a^{\beta+1}}{\sqrt{\frac{2e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}{(\beta+1)(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)^2}(2+2\beta)}}$$

$$C_2 = -\frac{\sqrt{2}}{\sqrt{\beta+1}} \sinh^{-1} \frac{a^{\beta+1}}{\sqrt{\frac{4e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}{(e^{\Psi(\beta+1)\sqrt{2}} - 1)^2}}}$$

$$C_2 = -\frac{\sqrt{2}}{\sqrt{\beta+1}} \sinh^{-1} \frac{(a^{\beta+1})(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)}{\sqrt{4e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}} \qquad (48)$$

Therefore (43) becomes

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \sinh^{-1} \frac{R^{\beta+1}(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)}{\sqrt{4e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}} - \frac{\sqrt{2}}{\sqrt{\beta+1}} \sinh^{-1} \frac{a^{\beta+1}(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)}{\sqrt{4e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}}$$

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \left(\sinh^{-1} \frac{R^{\beta+1}(e^{\Psi\sqrt{\beta+1}\sqrt{2}} - 1)}{\sqrt{4e^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}} A}} - \right.$$

sinh*−1aβ+1(eΨβ+12−1)4eΨβ+12 A*
(49)

Setting

$$S = e^{\Psi\sqrt{2(\beta+1)}} - 1$$

$$T = Ae^{\frac{\Psi\sqrt{\beta+1}}{\sqrt{2}}}$$

Equation (4.15) now becomes

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \left(\sinh^{-1} \frac{R^{\beta+1}S}{\sqrt{4T}} - \sinh^{-1} \frac{a^{\beta+1}S}{\sqrt{4T}} \right)$$

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \left\{ \ln\left(\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right)^2 + 1}\right) \right) - \ln\left(\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right)^2 + 1}\right) \right) \right\}$$

Simplifying further, we obtain

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \ln \frac{\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right)^2 + 1}\right)}{\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right)^2 + 1}\right)}$$

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \ln \frac{\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\frac{\left(R^{\beta+1}S\right)^2 + 4T}{4T}}\right)}{\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right) + \left(\sqrt{\frac{\left(a^{\beta+1}S\right)^2 + 4T}{4T}}\right)}$$

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \ln \frac{\left(\frac{R^{\beta+1}S}{2\sqrt{T}}\right) + \left(\frac{\sqrt{\left(R^{\beta+1}S\right)^2 + 4T}}{2\sqrt{T}}\right)}{\left(\frac{a^{\beta+1}S}{2\sqrt{T}}\right) + \left(\frac{\sqrt{\left(a^{\beta+1}S\right)^2 + 4T}}{2\sqrt{T}}\right)}$$

$$g(R) = \frac{\sqrt{2}}{\sqrt{\beta+1}} \ln \frac{R^{\beta+1}S + \sqrt{\left(R^{\beta+1}S\right)^2 + 4T}}{a^{\beta+1}S + \sqrt{\left(a^{\beta+1}S\right)^2 + 4T}} \quad (50)$$

$$g_R = \frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}} \left\{ \frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T} \right\} \quad (51)$$

Now with g(R) and g_R given in (51) and (52)

The stress components can be expressed in terms of g_R

Using equ (51) in (25), we have;

$$\sigma_{rr} = -P + \frac{\mu J_m}{J_m - (rg_R)^2}$$

$$\sigma_{rr} = -P + \frac{\mu J_m}{J_m - \left(r\frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}}\left\{\frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T}\right\}\right)^2} \quad (52)$$

Inserting equ (51) in equ(26), we have

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m((rg_R)^2 + 1)}{J_m - (rg_R)^2}$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m\left(\left(r\frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}}\left\{\frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T}\right\}\right)^2 + 1\right)}{J_m - \left(r\frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}}\left\{\frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T}\right\}\right)^2} \quad (53)$$

Inserting (51) in (27), we have;

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu J_m(rg_R)}{J_m - (rg_R)^2}$$

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu J_m\left(r\frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}}\left\{\frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T}\right\}\right)}{J_m - \left(r\frac{(\beta+1)\sqrt{2}}{\sqrt{\beta+1}}\left\{\frac{R^{\beta}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + R^{2\beta+1}S^2}{R^{\beta+1}S\sqrt{\left(R^{\beta+1}S\right)^2 + 4T} + \left(R^{\beta+1}S\right)^2 + 4T}\right\}\right)^2} \quad (54)$$

Recall that $\beta = \left(\frac{J_m}{2} - 1\right)$ from equ(34) and r = R from equa (11);

Equ (50) becomes

$$g(R) = \frac{2}{\sqrt{J_m}} \ln \frac{R^{\frac{J_m}{2}}S + \sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T}}{a^{\frac{J_m}{2}}S + \sqrt{\left(a^{\frac{J_m}{2}}S\right)^2 + 4T}} \tag{55}$$

and (51) becomes;

$$g_R = \frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}-1}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m-1}S^2}{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}}S\right)^2 + 4T} \right\} \tag{56}$$

Equations (52) – (54) are expressed thus

$$\sigma_{rr} = -P + \frac{\mu J_m}{\left(J_m - \left(R\frac{J_m}{\sqrt{J_m}}\left\{\frac{R^{\frac{J_m}{2}-1}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m-1}S^2}{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}}S\right)^2 + 4T}\right\}\right)\right)^2}$$

$$\sigma_{rr} = -P + \frac{\mu J_m}{\left(J_m - \left(\frac{J_m}{\sqrt{J_m}}\left\{\frac{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m}S^2}{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m}S^2 + 4T}\right\}\right)\right)^2}$$

$$\sigma_{rr} = -P + \frac{\left(R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m}S^2 + 4T\right)^2}{8T\left(R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m}S^2 + 2T\right)} \tag{57}$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m\left(R\frac{J_m}{\sqrt{J_m}}\left\{\frac{R^{\frac{J_m}{2}-1}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m-1}S^2}{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}}S\right)^2 + 4T}\right\}\right)^2 + \mu J_m}{\left(J_m - \left(R\frac{J_m}{\sqrt{J_m}}\left\{\frac{R^{\frac{J_m}{2}-1}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + R^{J_m-1}S^2}{R^{\frac{J_m}{2}}S\sqrt{\left(R^{\frac{J_m}{2}}S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}}S\right)^2 + 4T}\right\}\right)\right)^2}$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m \left(\frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T} \right\} \right)^2 + \mu J_m}{J_m - \left(\frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T} \right\} \right)^2}$$

$$\sigma_{\theta\theta} = -P + \frac{\mu J_m \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 \right)^2 + \mu \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T \right)^2}{8T \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 2T \right)} \tag{58}$$

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu J_m R \frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}-1} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m-1} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}} S\right)^2 + 4T} \right\}}{J_m - \left(R \frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}-1} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m-1} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + \left(R^{\frac{J_m}{2}} S\right)^2 + 4T} \right\} \right)^2}$$

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu J_m \frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T} \right\}}{J_m - \left(\frac{J_m}{\sqrt{J_m}} \left\{ \frac{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2}{R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T} \right\} \right)^2}$$

$$\sigma_{\theta r} = \sigma_{r\theta} = -P + \frac{\mu \left\{ \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 \right) \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 4T \right) \right\}}{8T \sqrt{J_m} \left(R^{\frac{J_m}{2}} S \sqrt{\left(R^{\frac{J_m}{2}} S\right)^2 + 4T} + R^{J_m} S^2 + 2T \right)} \tag{59}$$

Graph showing the relationship between the material parameter and displacement for radius 1.5 < R/a < 3.5

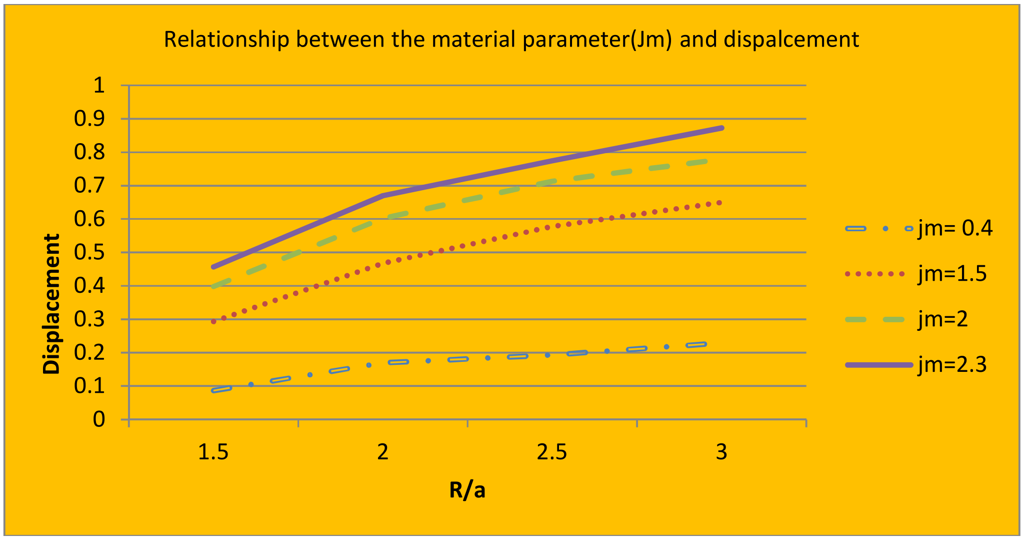

Fig. 1. Relationship between the material parameter and displacement

5. CONCLUSION

The material parameter for hardening and softening from the Gent's strain energy function is considered in this research work and its relationship with displacement and stresses is achieved for an incompressible hollow cylindrical tube. The nonlinear second order differential equation derived which is the common behaviour of elastic material is solved with the aid of software.

In this work, the analytical solution of displacement and stresses for an incompressible hollow cylindrical tube under azimuthal shear deformation are obtained. It is seen that as the change in the material parameter increases the displacement also increases likewise stresses. It is seen that changes on the material parameter affects the displacement and stresses.

It is seen that as the change in the material parameter increases the displacement also increases likewise stresses. It is seen that changes on the material parameter affects the displacement and stresses.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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