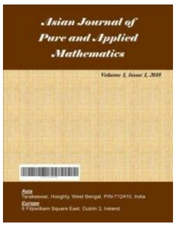

Asian Journal of Pure and Applied Mathematics

4(1): 766-773, 2022; Article no.AJPAM.1129

Fixed Point Theorems in Complete Metric Space Using E.A. and (CLR) Property

Manoj Kumar [a*] and Deepika [a]

[a] *Baba Mastnath University, Asthal Bohar, Rohtak, India.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Original Research Article

Received: 18 August 2022
Accepted: 21 October 2022
Published: 21 November 2022

Abstract

In this paper, we shall prove some common fixed point theorems for four weakly compatible self-maps along with E.A and (CLR) property in metric space.

Keywords: Fixed point; weakly compatible maps; E.A. property and (CLR) property.

2010 MSC: 47H10, 54H25.

1 Introduction

Fixed point is a point which remains invariant under the self map. Fixed point theory has wide applications in Economics, Biology, Game Theory etc. It is used to solve differential equations, fractional differential equations, integral equations etc [1,2]. In 1922, Banach [3] gave a method to evaluate a fixed point of a self map. Since then, researchers had generalized the Banach Contraction Principle in many ways to obtain new fixed point theorems (see [4-9],). In 2002, Aamri and Moutawakil [10] gave E.A. property to get new fixed point theorems. In 2011, Sintunavarat and Kuman [11] weaken the E.A. property and gave new called (CLR). The researchers used these properties in different ways to obtain new fixed point theorems [12-14]. In this paper we shall also used E.A. and (CLR) in metric space and give a new contraction to get the fixed point theorems.

2 Preliminaries

For the proof of our main results, we have the requirements of basic definitions from literature, which are as follows:

**Corresponding author: E-mail: manojantil18@gmail.com;*

Definition 2.1: A metric space (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 2.2 ([11]): Let f and g be two self-maps of a metric space (X, d), then they are said to satisfy (CLR_g) property if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gx$ for some $x \in X$.

Definition 2.3([8]): Recently, Jungck defined weakly compatible maps as follows:

Let A and S be mappings on a metric space (X, d) into itself. A and S are said to be weakly compatible if they commute at their coincidence points, that is, $Ax = Sx$, for some $x \in X$ implies that $ASx = SAx$.

Definition 2.4 ([10]): Two self mappings S and T of a metric space are said to satisfy E.A. property if there exists a sequence $\{x_n\} \in X$ such that

$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x_0$, for some $x_0 \in X$.

3 Main Results

Lemma 3.1([15]) : Let ϕ be the class of all functions ϕ: $[0,\infty)^6 \to \mathbb{R}$ having the following properties:

(ϕ_1): $\phi(u, v, v, u, u + v, 0) \leq 0$ or $\phi(u, v, u, u, 0, u + v) \leq 0$ for all $v > 0$ implies that $u < v$ and $v = 0$ implies that u = 0.

(ϕ_2): ϕ is non-decreasing in variables t_5 and t_6.
(ϕ_3): $\phi(u, u, 0, 0, u, u) \leq 0$ implies that $u = 0$.
(ϕ_3): ϕ is continuous in each coordinate variable.

Theorem 3.2: Let (X, d) be a metric space and A, B, S and T be self-maps of a metric space having the conditions:

(3.2) $AX \subset TX, BX \subset SX$.
(3.3) Either the pair (A, S) satisfies the (CLR_S) property or the pair (B, T) satisfies the (CLR_T) property.
(3.4) The pairs (A, S) and (B, T) are weakly compatible.
(3.5) $\phi\begin{pmatrix} d(Ax,By), d(Sx,Ty), d(Sx,Ax), \\ d(By,Ty), d(Sx,By), d(Ty,Ax) \end{pmatrix} \leq 0$,

$\forall\, x, y \in X$ and $\phi \in \Phi$.

Then A, B, S and T have a unique fixed point in X.

Proof: Let (B, T) satisfies the (CLR_T) property, then $\exists$ a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = Tx, \text{ for some } x \text{ in } X. \tag{3.6}$$

From (3.2),

$$BX \subset SX, \text{ then there exists a sequence } \{y_n\} \text{ in } X \text{ such that } Bx_n = Sy_n. \tag{3.7}$$

From (3.6), we get

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Tx.$$

Now, we will prove that $\lim_{n\to\infty} Ay_n = Tx$.
For this, let $x = y_n$ and $y = x_n$ in (3.5), we get

$$\phi\begin{pmatrix} d(Ay_n,Bx_n), d(Sy_n,Tx_n), d(Sy_n,Ay_n), \\ d(Bx_n,Tx_n), d(Sy_n,Bx_n), d(Tx_n,Ay_n) \end{pmatrix} \le 0. \tag{3.8}$$

Now from (3.7),

$$\phi\begin{pmatrix} d(Ay_n,Bx_n), d(Bx_n,Tx_n), d(Bx_n,Ay_n), \\ d(Bx_n,Tx_n), d(Bx_n,Bx_n), d(Tx_n,Ay_n) \end{pmatrix} \le 0.$$

Taking limit as $n \to \infty$, we get

$$\phi\begin{pmatrix} d(Ay_n,Tx), d(Tx,Tx), d(Tx,Ay_n), \\ d(Tx,Tx), d(Tx,Tx), d(Tx,Ay_n) \end{pmatrix} \le 0.$$
$$\Rightarrow \phi\begin{pmatrix} d(Ay_n,Tx), 0, d(Tx,Ay_n), \\ 0, 0, d(Tx,Ay_n) \end{pmatrix} \le 0.$$

From above Lemma 3.1, $d(Ay_n, Tx) = 0$.

$$\Rightarrow Ay_n = Tx.$$
$$\Rightarrow \lim_{n\to\infty} Ay_n = Tx. \tag{3.9}$$

As given, S is mapping of X into itself, therefore $SX \subset X$.

Hence, $Tx = Sh$, for some $h \in X$.

Thus, we have $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = Tx = Sh$. (3.10)

Now, letting $x = h$ and $y = x_n$ in (3.5), we get

$$\phi\begin{pmatrix} d(Ah,Bx_n), d(Sh,Tx_n), d(Sh,Ah), \\ d(Bx_n,Tx_n), d(Sh,Bx_n), d(Tx_n,Ah) \end{pmatrix} \le 0.$$

Taking limit as $n \to \infty$ and using (3.10), we get

$$\phi\begin{pmatrix} d(Ah,Sh), d(Sh,Sh), d(Sh,Ah), \\ d(Sh,Sh), d(Sh,Sh), d(Sh,Ah) \end{pmatrix} \le 0.$$
$$\Rightarrow \phi\begin{pmatrix} d(Ah,Sh), 0, d(Sh,Ah), \\ 0, 0, d(Sh,Ah) \end{pmatrix} \le 0.$$

From the above Lemma 3.1,

$$d(Ah,Sh) = 0.$$
$$\Rightarrow Ah = Sh. \tag{3.11}$$

From (3.4), we have

A and S are weakly compatible, therefore $ASh = SAh$.

Hence, $ASh = AAh = SAh = SSh$. (3.12)

From (3.2), $AX \subset TX$, then there exists some $q \in X$ such that $Ah = Tq$. (3.13)
Similarly, we can show that $Tq = Bq$.

From (3.13), $Tq = Ah$.

Therefore, $Tq = Bq$ which implies that $Ah = Sh = Tq = Bq$. (3.14)

From (3.4), B and T are weakly compatible, therefore

$BTq = BBq = TBq = TTq$. (3.15)

Moreover, we will prove that $AAh = Ah$.

Let if possible, $AAh \neq Ah$.

Putting $x = Ah$ and $y = q$ in (3.5),

$$\phi\begin{pmatrix} d(AAh, Bq), d(SAh, Tq), d(SAh, AAh), \\ d(Bq, Tq), d(SAh, Bq), d(Tq, AAh) \end{pmatrix} \leq 0.$$

Now using (3.12) and (3.14),

$$\Rightarrow \phi\begin{pmatrix} d(AAh, Ah), d(AAh, Ah), d(AAh, AAh), \\ d(Ah, Ah), d(AAh, Ah), d(Ah, AAh) \end{pmatrix} \leq 0.$$
$$\Rightarrow \phi\begin{pmatrix} d(AAh, Ah), d(AAh, Ah), 0, \\ 0, d(AAh, Ah), d(Ah, AAh) \end{pmatrix} \leq 0.$$
$$\Rightarrow d(AAh, Ah) = 0.$$
$$\Rightarrow AAh = Ah.$$

Consequently, we can prove that

$$BAh = Ah, TAh = Ah, SAh = Ah.$$

Hence, Ah is a common fixed point of A, B, S and T.

Uniqueness: Let m and n be two fixed points of A, B, S and T.

So, $Am = Bm = Sm = Tm = m$ and $An = Bn = Sn = Tn = n$.

Now putting $x = m, y = n$ in (3.5), we get

$$\phi\begin{pmatrix} d(Am, Bn), d(Sm, Tn), d(Sm, Am), \\ d(Bn, Tn), d(Sm, Bn), d(Tn, Am) \end{pmatrix} \leq 0.$$
$$\Rightarrow \phi\begin{pmatrix} d(m, n), d(m, n), d(m, m), \\ d(n, n), d(m, n), d(n, m) \end{pmatrix} \leq 0.$$
$$\Rightarrow d(m, n) = 0.$$
$$\Rightarrow m = n.$$

Hence, A, B, S and T have a common fixed point which is unique.

Theorem 3.3: Let (X, d) be a metric space and A, B, S and T be self-maps of a metric space having the conditions:

(3.16) $AX \subset TX, BX \subset SX$.
(3.17) the pairs (A, S) and (B, T) are weakly compatible.
(3.18) $\phi\begin{pmatrix} d(Ax,By),d(Sx,Ty),d(Sx,Ax), \\ d(By,Ty),d(Sx,By),d(Ty,Ax) \end{pmatrix} \leq 0$,
$\forall\, x, y \in X$ and $\phi \in \Phi$.
(3.19) one of AX, BX, SX or TX is a complete subspace of X.
(3.20) either the pair (A, S) or the pair (B, T) satisfies the E.A. property.

Then A, B, S and T have a unique fixed point in X.

Proof: Let (B, T) satisfies the E. A. property, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = x_0, \text{ for some } x_0 \text{ in } X. \tag{3.21}$$

From (3.16), $BX \subset SX$, then there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. (3.22)

From (3.21), we get

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = x_0.$$

Now, we will prove that $\lim_{n\to\infty} Ay_n = x_0$.

For this, let $x = y_n$ and $y = x_n$ in (3.18), we get

$$\phi\begin{pmatrix} d(Ay_n,Bx_n),d(Sy_n,Tx_n),d(Sy_n,Ay_n), \\ d(Bx_n,Tx_n),d(Sy_n,Bx_n),d(Tx_n,Ay_n) \end{pmatrix} \leq 0.$$

Now from (3.22),

$$\phi\begin{pmatrix} d(Ay_n,Bx_n),d(Bx_n,Tx_n),d(Bx_n,Ay_n), \\ d(Bx_n,Tx_n),d(Bx_n,Bx_n),d(Tx_n,Ay_n) \end{pmatrix} \leq 0.$$

Taking limit as $n \to \infty$, we get

$$\phi\begin{pmatrix} d(Ay_n, x_0), d(x_0, x_0), d(x_0, Ay_n), \\ d(x_0, x_0), d(x_0, x_0), d(x_0, Ay_n) \end{pmatrix} \leq 0.$$
$$\Rightarrow \phi\begin{pmatrix} d(Ay_n, x_0), 0, d(x_0, Ay_n), \\ 0, 0, d(x_0, Ay_n) \end{pmatrix} \leq 0.$$

From above Lemma 3.1, $d(Ay_n, x_0) = 0$.

$$\Rightarrow Ay_n = x_0.$$
$$\Rightarrow \lim_{n\to\infty} Ay_n = x_0. \tag{3.23}$$

Assume that SX is a complete subspace of X.

Then, $x_0 = Sh$, for some h in X.

Subsequently, we have

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Ay_n = x_0 = Sh. \tag{3.24}$$

Now, letting $x = h$ and $y = x_n$ in (3.18), we get

$$\phi\begin{pmatrix} d(Ah, Bx_n), d(Sh, Tx_n), d(Sh, Ah), \\ d(Bx_n, Tx_n), d(Sh, Bx_n), d(Tx_n, Ah) \end{pmatrix} \le 0.$$

Taking limit as $n \to \infty$ and using (3.24), we get

$$\phi\begin{pmatrix} d(Ah, Sh), d(Sh, Sh), d(Sh, Ah), \\ d(Sh, Sh), d(Sh, Sh), d(Sh, Ah) \end{pmatrix} \le 0.$$

$$\Rightarrow \phi\begin{pmatrix} d(Ah, Sh), 0, d(Sh, Ah), \\ 0, 0, d(Sh, Ah) \end{pmatrix} \le 0.$$

From the above Lemma 3.1,

$$d(Ah, Sh) = 0.$$

$$\Rightarrow Ah = Sh. \tag{3.25}$$

From (3.17), we have

A and S are weakly compatible, therefore $ASh = SAh$.

Hence, $ASh = AAh = SAh = SSh$. (3.26)

From (3.16), $AX \subset TX$, then there exists some $q \in X$ such that $Ah = Tq$. (3.27)

Similarly, we can show that $Tq = Bq$.

From (3.27), $Tq = Ah$.

Therefore, $Tq = Bq$ which implies that $Ah = Sh = Tq = Bq$. (3.28)

From (3.17), B and T are weakly compatible, therefore

$$BTq = BBq = TBq = TTq. \tag{3.29}$$

Moreover, we will prove that $AAh = Ah$.

Let if possible, $AAh \neq Ah$.

Putting $x = Ah$ and $y = q$ in (3.18),

$$\phi\begin{pmatrix} d(AAh, Bq), d(SAh, Tq), d(SAh, AAh), \\ d(Bq, Tq), d(SAh, Bq), d(Tq, AAh) \end{pmatrix} \le 0.$$

Now using (3.26) and (3.28),

$$\Rightarrow \phi\begin{pmatrix} d(AAh, Ah), d(AAh, Ah), d(AAh, AAh), \\ d(Ah, Ah), d(AAh, Ah), d(Ah, AAh) \end{pmatrix} \le 0.$$

$$\Rightarrow \phi\begin{pmatrix} d(AAh, Ah), d(AAh, Ah), 0, \\ 0, d(AAh, Ah), d(Ah, AAh) \end{pmatrix} \le 0.$$

$$\Rightarrow d(AAh, Ah) = 0.$$

$\Rightarrow AAh = Ah.$

Consequently, we can prove that

$BAh = Ah, TAh = Ah, SAh = Ah.$

Hence, Ah is a common fixed point of A, B, S and T.

Uniqueness: Let p and q be two fixed points of A, B, S and T.

So, $Ap = Bp = Sp = Tp = p$ and $Aq = Bq = Sq = Tq = q$.

Now putting $x = p, y = q$ in (3.5), we get

$$\phi\begin{pmatrix} d(Ap, Bq), d(Sp, Tq), d(Sp, Ap), \\ d(Bq, Tq), d(Sp, Bq), d(Tq, Ap) \end{pmatrix} \leq 0.$$
$$\Rightarrow \ \phi\begin{pmatrix} d(p, q), d(p, q), d(p, p), \\ d(q, q), d(p, q), d(q, p) \end{pmatrix} \leq 0.$$
$$\Rightarrow d(p, q) = 0.$$
$$\Rightarrow p = q.$$

Hence, A, B, S and T have a common fixed point which is unique.

Example 3.4: Let (X, d) be a metric space with $d(x, y) = |x - y|$, where $X = [0, 1]$.

Define the self-maps A, B, S and T on X by

$$Ax = Bx = \frac{1}{2}, Sx = \frac{x+2}{5}, Tx = \frac{6x-1}{6}, \text{ for all } x \in X.$$

Therefore, $d(Ax, Bx) = \left|\frac{1}{2} - \frac{1}{2}\right| = 0.$

$$d(Sx, Tx) = \left|\frac{x+2}{5} - \frac{6x-1}{6}\right| = \left|\frac{24x-17}{30}\right|.$$

Hence, $0 = d(Ax, Bx) \leq d(Sx, Tx)$, for every x in X. If we define $\phi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - t_2$, it is easy to see that all the conditions of Theorem 3.2 as well as Theorem 3.3 hold and there exists a unique $x = \frac{1}{2}$ such that

$$A\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = T\left(\frac{1}{2}\right) = \frac{1}{2}.$$

4 Conclusion

In the present paper, some fixed point theorems for two pairs of weakly compatible maps along with E.A. property and (CLR) property are proved.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] George A, Veeramani P. On some results in fuzzy metric spaces. Fuzzy Sets and Systems. 1994;64: 395-399.

[2] Singh B, Jain S. Weak compatibility and fixed point theorems in fuzzy metric spaces. Ganita. 2005;56(2):167-176.

[3] Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations itegrale. Fundam. Math. 1922;3:133-181.

[4] Ahmed MA. Common fixed point theorems for weakly compatible mappings. Rocky Mountain J. Math. 2003;33(4):1189-1203.

[5] Chugh R, Kumar S. Common fixed points for weakly compatible maps. Proc. Indian Acad. Sci. Math. Sci. 2001;111(2):241-247.

[6] Ciric Lj. B, Ume JS. Some common fixed point theorems for weakly compatible mappings. J. Math. Anal. Appl. 2006;314(2):488-499.

[7] Jungck G, Rhoades BE. Fixed points for set valued functions without continuity. Indian J. Pure Appl. Math. 1998;29(3):227-238.

[8] Jungck G. Common fixed points for non-continuous non-self maps on non metrics paces. Far East J. Math. 1996;4(2):199-215.

[9] Popa V. A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation. Filomat. 2005;19:45-51.

[10] Aamri M, Moutawakil D. El. Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. 2002;270(1):181-188.

[11] Sintunavarat W, Kuman P. Common fixed point theorems for a pair of weakly compatible mappings in Fuzzy Metric Spaces. Journal of Applied Mathematics. 2011;Article ID 637958:14.

[12] Rasham T, Shoaib A, Alshoraify S, Park C, Lee JR. Study of multivalued fixed point theorems for generalized contractions in double controlled dislocated quasi metric type spaces. AIMS Mathematics. 7(1):1058-1073.

[13] Rasham T, Asif A, Aydi H, La Sen M. De. On pairs of fuzzy dominated mappings and applications. Advances in Difference Equations. 2021;1:1-22.

[14] Rasham T, Shoaib A, Park C, Sen MDL, Aydi H, Lee JR. Multivalued fixed point results for two families of mappings in modular like metric spaces with applications. Complexity. 2020;2020:1-10.

[15] Sedghi S, Shobe N. Common fixed point theorems for four mappings in complete metric spaces. Bulletin of the Iranian Mathematical Society. 2007;33(2):37-47.
