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On Analytical Review of Conjugate and Reflexive Spaces

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

In this review, we started by considering a normed space X, its dual and bi-dual spaces. We define functionals on X, as those functionals that are bounded linear functionals and then establish a corresponding unique bounded linear functional that defines a canonical mapping of X into X^{**} such that C is linear, injective and preserves norm. By this, isomorphism is established so that reflexive maps are realized. In the cause of this work, we invoked the idea of compact topological space and operators mapped on them. We also discussed separable spaces as it relates to this topic. The work was rounded up in section three by discussing the Tychonoff's theorem as it relates to the weak topology in conjugate spaces, adjoint operators and conjugate spaces of L^{∞} and $C[0,1]$.

Keywords: Normal space; dual space; functional; reflexive space; conjugate space; isomorphism.

1. INTRODUCTION

In this section, we discuss the reflexivity of normed spaces but let us first recall that a vector space X is said to be algebraically reflexive if the canonical mapping $C: X \to X^{**} = (X^*)^*$ is the second algebraic dual space of X and the mapping C is defined by $x \to g_x$

where

$$g_x(f) = f(x) \qquad (f \in X' \text{variable}) \qquad (1.1)$$

That is for any $x \in X$ the image is linear functional g_x defined by (1.1). If X is finite dimensional, then X is algebraically reflexive. Now on our actual task, we consider a normed space X, its dual space and moreover, the dual space $(X^*)^*$ of X^*. This space is denoted by X^{**} and is called the second dual space of X (or bi-dual space of X).

We define a functional g_x on X^* by choosing a fixed $x \in X$ and setting

$$g_x(f) = f(x) \qquad (f \in X' \text{variable}) \qquad (1.2)$$

This looks like (1.1), but note that now f is bounded and g_x turns out to be bounded, too since we have the basic.

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Lemma 1.1. Norm of g_x[1]: For every fixed x in a normed space X, the functional g_x defined by (1.2) is a bounded linear functional on X^*, so that $g_x \in X''$ and has the norm

$$\|g_x\| = \|x\| \qquad (1.3)$$

Proof: Linearity of g_x is obvious and (1.3) follows from (1.2) and

$$\|g_x\| = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{\|g_x f\|}{\|f\|} = \sup_{\substack{f \in X' \\ f \neq 0}} \frac{\|f(x)\|}{\|f\|} = \|x\|$$

To every $x \in X$ there corresponds a unique bounded linear functional $g_x \in X''$ given by (1.2). This defines a mapping

$$C: X \to X'', \quad x \to g_x \qquad (1.4)$$

$C$is called the canonical mapping of X into X^{**}. We show that C is linear and injective and preserves the norm. this can be expressed in terms of an isomorphism of normed spaces.

Lemma1.2. Canonical Mapping [2]: The canonical mapping C given by (1.4) is an isomorphism of the normed space X onto the normed space $\Re(C)$ the range C.

Proof: Linearity of C is seen below because

$$g_{ax+\beta y}(f) = f(ax+\beta y) = af(x) + \beta f(y) a g_x(f) + \beta g_y(f)$$

In particular, $g_x - g_y = g_{x-y}$, hence, by (1.3) we obtain

$$\|g_x - g_y\| = \|g_{x-y}\| = \|x - y\|$$

This shows that C is isometric, it preserves the norm. Isometry implies injectivity. We can also see this directly from our formula. Indeed if $x \neq y$, then $g_x \neq g_y$, hence C is bijective and regarded as a mapping onto its range. $X$is said to be embeddable in a normed space Z if X is isomorphic with a subspace of Z, but note that here we are dealing with isomorphisms of normed spaces, that is vector space isomorphisms which preserve norm.

Definition 1.1. Reflexivity [3]: A normed space X is said to be reflexive if

$$\Re(C) = X^{**}$$

Where $C: X \to X^{**}$ is the canonical mapping given by (1.4) and (1.2).

If X is reflexive, it is isomorphic (hence isometric) with X^{**}. It is interesting that the converse does not generally hold.

Furthermore, completeness does not imply reflexivity, but conversely we have

Theorem 1.4. Completeness [4]: If a normed space X is reflexive, it is complete (hence a Banach space).

Proof: Since X^{**} is dual space X^*, it is complete reflexivity of X means that $\Re(C) = X''$. Completeness of X now follows from that X''. $R^n$is reflexive. This follows directly, it is typical of any finite dimensional normed space X. Indeed, if $\dim X < \alpha$, then every linear functional on X is bounded, so that $X' = X^*$and algebraic reflexivity of X thus implies.

Theorem 1.5. Finite Dimension [5]: Every finite dimensional normed space is reflexive. L^p with $1 < p < +\infty$ is reflexive. Similarly, $L^p[a,b] < p < +\infty$ is reflexive as can be shown. It can also be proved that non-reflexive space are $C[a,b], l', L'[a,b], l^\infty$ and the space of all convergent sequences of scalars and C_0 is the space of all sequences of scalars converging to zero.

Theorem 1.6. Hilbert Space [6]: Every Hilbert space H is reflexive.

Proof: We shall prove surjectivity of the canonical mapping $C: H \to H^{**}$ by showing that for every $g \in H''$, there is an $x \in H$ such that $g = C_x$. As a preparation we define $A: H^* \to H^{**}$by $Af = z$, where z is given by Riesz representation $f(x) = (x, z)$. We know that A is bijective and isometric. A is conjugate linear, as we see from now H' is complete and a Hilbert space with inner product defined by

$$\langle f_1, f_2 \rangle = \langle Af_2, Af_1 \rangle$$

Note: The order f_1, f_2 on the both sides. (IPI) to (IP4) is readily verified. In particular, (IP2) follows from the conjugate linearity of A:

$$\langle af_1, f_2 \rangle 1 = \langle Af_2, A(af_1) \rangle = \langle Af_2, aAf_1 \rangle = a\langle f_1, f_2 \rangle 1$$

Let $g \in H''$ be arbitrary, let its Riesz representation be

$g\langle f\rangle = \langle f, f_0\rangle_1 = \langle Af_0, Af\rangle$

We now remember that $f(x) = \langle x, z\rangle$, where $z = Af$. Writing $Af_0 = x$, we thus have

$\langle Af_0, Af\rangle = \langle x, z\rangle = f(x)$

Together, $g(f) = x$, that is $g = Cx$ by the definition of C. Since $g \in H^{**}$ was arbitrary, C is surjective, so that H is reflexive.

Hence, if a normed space X is reflexive, X^{**} is isomorphic with X, so that in case, separability of X implies separability of X^{**} and the space X^{*} is also separable. From this we have the following result.

A seperable normed space X with a non-separable dual space X^{*} cannot be reflexive

Example 1.1: L is not reflexive

Proof:L is separable, but $L^1 = L^\infty$ is not; and the desired theorem (1.8) will be obtained from the following lemma. A simple illustration of the lemma is shown in Fig. 1.

Lemma 1.7. Existence of a functional [7]: Let Y be a proper closed subspace of a normed space X. Let $x_0 \in X \to Y$ be arbitrary and

$\delta = \inf_{\tilde{y} \in Y} \|\tilde{y} - x_0\|$ (1.5)

The distance from $x_0$to Y, then there exists an $f \in X'$ such that

$\|\tilde{f}\| = 1, \quad \tilde{f}(y) = 0$for all $y \in Y, \quad \tilde{f}(x_0) = \delta$ (1.6)

Proof: The idea of the proof is simple, we consider the subspace $Z \subset X$ spanned $Y$and x_0, define on Z a bounded linear functional f by

$f(z) = f(y + ax_0) = a\delta \qquad y \in Y$ (1.7)

Satisfies (1.6) and extends f to X by (1.2). The details are as follows:

Every $z \in Z = span\ (Y \cup \{x_0\})$ has a unique representation

$Z = y + ax_0 = a\delta \qquad y \in Y$

This is used in (1.7). Linearity of f readily seen, also since Y is closed, $\delta > 0$, so that $f \neq 0$. Now $a = 0$ gives $f(y) = 0$ for all $y \in Y$ for

To illustrate lemma (1.4) for the Euclidean space $X = R^3$, where Y is represented by $\xi^2 = \xi_1/2, \xi_3 = 0, x_0 = (1.3,0)$, so that $\sqrt{5}$, $Z = SPAN\ (y \cup \{x_0\})$ is the $\xi_1, \xi_2 -$plane and $f(z) = 9 - \xi_1 + 2\xi_2/\sqrt{5}$, $\alpha = 1$ and $y = 0$ we have $f(x) = \delta$.

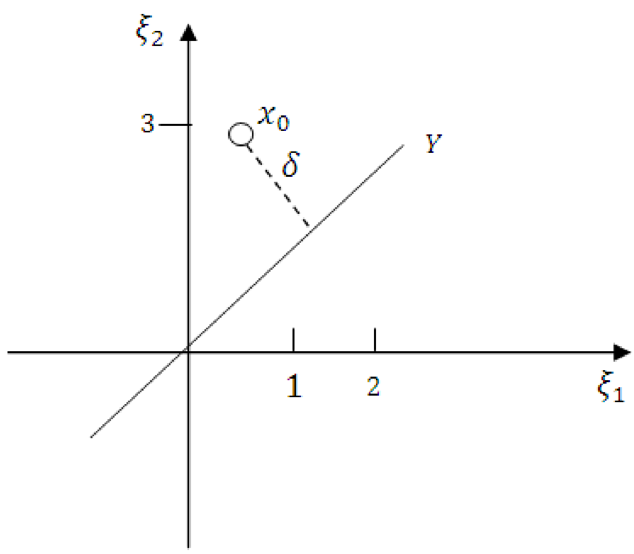

Fig. 1. Linearity of the functional "f"

We show that f is bounded $\alpha = 0$gives $f(z) = 0$. Let $\alpha \neq 0$, using (1.5) and noting that $-(1/\alpha)y \in Y$, we obtain

$$|f(z)| = |\alpha| \inf_{y \in Y} \|\tilde{y} - x_0\| \leqq |\alpha| \left\| -\frac{1}{\alpha} y - x_0 \right\|$$
$$= \|y + \alpha x_0\|$$

that is $|f(z)| \leq \|z\|$. Hence, f is bounded and $\|f\| \leq 1$. We show that $\|f\| \geq 1$. By the definition of an infimum, Y contains a sequence (y_n) such that $\|y_n - x_0\| \to \delta$.

Let $z_n = y_n - x_0$, then we have $f(z_n) = -\delta$ by (1.8) with $\alpha = -1$.

Also,

$$\|f\| = \sup_{\substack{z \in Z \\ z \neq 0}} \frac{|f(z)|}{\|z\|} \geqq \frac{|f(z_n)|}{\|z_n\|} \frac{\delta}{\|z_n\|} \to \frac{\delta}{\delta} = 1$$

As $n \to \alpha$. Hence, $\|f\| \geqq 1$, so that $\|f\|$ by the Hahn-Banach theorem for normed spaces we can extend f to X without increasing theorem for normed spaces we can extend f to X without increasing the norm. Using this lemma, we shall obtain the desired.

Theorem 1.8. Separability [3]: if the dual spaces X^* of a normed space X is separable, then X itself is separable.

Proof: We assume that X^* is separable. Then the unit sphere $U^* = \{f \| f\| = 1\} \subset X'$ also contain a countable dense subset, say (f_n). Since $f_n \in U^*$, we have

$$\|f_n\| = \sup_{\|x\|=1} |f_n(x)| = 1$$

By the definition of a supremum we can find points $x_n \in X$ of norm 1 such that

$$|f_n(x_n)| \geqq \frac{1}{2}$$

Let Y be the closure of span (x_n). Then Y is separable because Y has a countable dense subset, namely, the set of all linear combinations of the x_n's with coefficients whose real and imaginary parts are rational.

We show that $Y = X,$ suppose $Y \neq X$. Then since Y is cosed by lemma (1.6) there exists an $f \in X'$ with $\|f\| = 1$ and $f(y) = 0$ for all $y \in Y$, we have $f(x_n) = 0$ and for all n

$$\frac{1}{2} \leqq |f_n(x_n)| = |f_n(x_n) - \tilde{f}(x_n)| = |(f_n - \tilde{f})(x_n)|$$
$$\leqq \|f_n - \tilde{f}\| \|x_n\|$$

where, $\|x_n\| = 1$. Hence, $\|f_n - \tilde{f}\| \geqq \frac{1}{2}$, but this contradicts the assumption that (f_n) is dense in U': in fact $\|\tilde{f}\| = 1$.

In the same way one shows that $k_y(z)$ has any number of derivatives and these are equal to the application of L_y to the corresponding derivatives of f_2. In particular, $\Delta k_y(z) = Ly(\Delta f_z)$. Since however, Δf_z is the function.

$\sum_{j-1}^{2} \delta^2 (\log|x - z|)\delta_z^2 = 0,$ we conclude in x that $k_y(z)$ is harmonic outside $\delta\Omega$.

Since $\log|x - z|$ is harmonic $x \in \Omega$ when $z \notin \bar{\Omega}$ we have

$ky(z) = Ly(fz) = \log|y - z|$if $z \notin \Omega$, we have

Now take a point z in Ω and near$\delta\Omega$. Denote by z' its reflection with respect to Tz. Then

$$\|f_z - \tilde{f}_z\| = \max_{x \in \delta\Omega} \log \frac{|z-x|}{|z-x|} \to 0 \text{if} z \to z_0, \ z_0 \in \delta\Omega$$

It follows that $L_y(f_z - f_z') \to 0$ that is

$$ky(z) - ky(z') \to 0 \text{if } z \to z_0.$$

But, by (1.8), $\lim ky(z^*)$ exists and equal $\log|y - z_0|$. Hence, $k_y(z)(z \in \Omega)$ can be extended into a continuous function $k_y(z)$ in Ω and $k_y(z) = \log|y - z|$ if $z \in \delta\Omega$. The function $-\log|x - y| + k_y(x)$ satisfies the condition (i)-(iii) in the definition of green's function.

2. CONJUGATE SPACES AND REFLEXIVE SPACES

2.1 Prelimnaries to Conjugate and Reflexive Spaces

Theorem 2.1.1: Let X be a normed linear space. ifx^* is separable, then X is separable.
The converse is not true in general.

Proof: Let $\{x_n^*\}$ be a dense sequence in X^* and choose $x_n \in X$ such that $\|x_n\| = 1, |x_n^*(x_n)| > \|x_n^*\|/2$. Denote by A the set of all linear combinations of the x_n with rational coefficients. $A$is countable and it suffices to show that it is dense in X. If $A \neq X$, then there exists an $x^* \neq 0$ such that $x^*(x) = 0$ for all $x \in A$. Let $\{x_n^*\}$ be a subsequence of $\{x_n^*\}$ that converges to x^*. Then,

$$\|X^* - X_n^*\| \geq |(X^* - X_n^*)(X_n)| = |X_n^*(X_n)| \geq \frac{\|X_n^*\|}{2}$$

Hence, $X_n^* \to 0$. It follows that $X^* = 0 - a$ contradiction

Definition 2.1.1. [8]: Let X and Y be normed linear spaces, suppose there exists one to one linear map T front X onto Y. Then we say that X and Y are linearly isomorphic (or briefly isomorphic) and we call T a linear isomorphism (or briefly an isomorphism). If further $\|T\| = \|X\|$ for all $x \in X$, then we say that X and Y are isometrically isomorphic and all we call T an isomorphism.

Consider now the conjugate of X^*, we denote it by X^{**}. To each $x \in X$ we can correspond an element $\hat{x}$ in X^{**} by $\hat{x}((X^*) = x^*(X)$ for all $x^* \in X$, we write $\hat{x} = k(X)$.

Theorem 2.1.2. [7]: Let X be a normed linear space, then the natural embedding from X into X^{**} is an isometric isomorphism between X and $\hat{X}$. The linearity of k is obvious , The relation $\|x\|$ follows.

The natural embedding of X into X^{**} is useful even for results concerning X and X^* only. We give such a result in the following theorem.

Theorem 2.1.3 [1]: Let $\{X_\alpha\}$ be a set of element in a normed linear space X. Suppose $X < \infty$ then for any $x^* \in X^*$, then $\sup|X_\alpha| < \infty$.

Proof: We can apply the principle of uniform boundedness to the family $\{\hat{x}_\alpha\}$. We conclude that $\sup_\infty \|\hat{X}_\alpha\| < \infty$, from this assertion follows

Definition 2.1.2 [9]: A banach space X is called reflexive if $k(X) = X^{**}$.

We shall prove later on that L^p spaces are reflexive if $1 < p < \infty$, but are not reflexive if $p = 1, p = \infty$.

Lemma 2.1.4 [10]: Let X and Y be isometrically isomorphic normed linears, if Y is reflexive then X is reflexive.

Proof: Let σ be the isometric isomorphism from Xonto Y. Define a map τ from X^* into Y^* by

$$(\tau x^*)(y) = x^*(\sigma^{-1} y)$$

It is easily seen that I is an isometric isomorphism from X^*onto Y^*. Similarly, define an isometric isomorphism ρ from X^{**} onto Y^{**} by

$$(\rho x^{**})(y) = x^{**}(\tau^{-1} y^*)$$

Denote by k_1 and k_2 the natural imbeddings of X into X^{**} and of Y onto Y^{**} respectively. Now take any element x_0^{**} in X^{**} and define $x_0 = \sigma^{-1} k_2^{-1} \rho x^{**}$. the point x_0 is in X and for any $x^* \in X^*$,

$$x^*(x_0) = x^*[\sigma^{-1}(k_2^{-1} \rho x_0^{**})] = (\tau x^*)(k - 1\, \rho x_0^{**})$$
$$= (\rho x_0^{**}) = \rho x^{**}(\tau x^*) = x_0^{**}(x^*)$$

Thus, $k_1 x_0 = x_0^{**}$. We have thus shown that $k_1(X) = X^{**}$, that is X is reflexive

Theorem 2.1.5 [11]: A closed linear subspace is reflexive

Proof: Let Y be closed linear subspace of a reflexive Banachspace X . Consider the map $\sigma: X^* \to Y^*$ defined by $(\sigma x^*)(y) = x^*(y)$for $y \in Y$. Note that $\|\sigma x^*\| \leq \|y^{**}\|$. Next define for $y^{**} \in Y^{**}, (\tau y^{**})(x^*) = y^{**}(\sigma x^*)$ for $x^* \in X^*$. Then $|(\tau y^{**})(x^*)| \leq \|y^{**}\| \|x^*\|$. Hence, $k^{-1}(\tau y^{**})$ is a well-defined element of X. We shall prove that it belongs to Y. If $x = k - 1(\tau y^{**}) \notin Y$, then there exists an $x^* \in X^*$ with $x^*(y) = 0$ for all $y \in Y$ and $x^*(x) = 0$. Since $x^*(y) = 0$ for all $y \in Y, \sigma x^* = 0$ hence,

$$0 = y^{**}(\sigma x^*) = (\tau y^{**})(x^*) = (kx)(x^*) = x^*(x)$$a contradiction

We have thus proved $k^{-1}[\tau(Y^{**})] \subset Y$. Now let $y^{**} \in Y^{**}$ and defined $x_0 = k_0^{-1}(\tau y^{**})$. For any $y^* \in Y^*$, let x^* be any extension of y^* to an element of X^*. Then $y^* = \sigma x^*$ and

$$y_0^{**}(y^*) = (\tau y_0^{**})(x^*) = (kx_0)(x^*) = x^*(x_0)$$
$$= y^*(x_0)$$

Since $x_0 \in Y$, then y_0^{**} is the image of x_0 under the natural imbedding of Y into Y^{**}. This proves that Y is reflexive.

Theorem 2.1.6 [7]: A Banach space is reflexive if and only if its conjugate is reflexive.

Proof: if X is reflexive, then the same is true also of X^{**}, which is then isometrically isomorphic to X (with the map k). It is obvious that $(X^*)^{**} = [(X^*)^*] = (X^{**})^*$. Now let $x_0^{***} \in (X^*)^{**}$. Then $x_0^{***} \in (X^{**})^*$, thus the functional $x_0^{***}k$ defined by $x^* = x_0^{***}k$ defined by $x^*(x_0) = x_0^{***}(kx)$ for $x \in X$ is in X^* and

$$x^{**}(x_0^*) = (kx)(x_0^*) = x_0^*(x) = x^{***}(kx) = x_0^{***}(x_0^{**})$$

for any $x^{**} \in X^{**}$. Hence, x_0^{**} is the image of x_0^* under the natural imbedding. Since x_0^{***} is arbitrary element of $(X^{**})^{**}$, we conclude that X^* is reflexive. Suppose conversely that X^* is reflexive. Then by what we have just proved X^{**} is reflexive. Since $k(X)$ is a closed linear subspace of X^{**}, theorem (2.1.5) shows that it is reflexive. Hence, by lemma (2.1.4) also X is reflexive. We now introduce an important concept of convergence.

Definition 2.1.3 [8]: Let X be a normed linear space, a sequence $\{X_n\}$ in X is said to be weakly convergent if there exists an element $x \in X$ such that $\lim x^*(x_n) = x^*(x)$ for any $x^* \in X^*$, we call x the weak limit of $\{X_n\}$ and we say that $\{X_n\}$ converges weakly to x. A set $k \subset X$ is called weakly sequentially compact if every sequence $\{X_n\}$ in K contains a subsequence that converges weakly to point in K. A sequence $\{X_n\}$ in X called a weak Cauchy sequence if $\{x^*(x_n)\}$ is a Cauchy sequence for any $x^* \in X^*$. The space X is called weakly complete if every weak Cauchy sequence has a weak limit. A set $K \subset X$ is weakly closed if the weak limit of any weakly convergent sequence $\{X_n\}$ in K is also in K.

Theorem 2.1.7 [12]: Let $\{X_n\}$ be a weakly convergent sequence in a normed linear space X, then:

(a) $\{X_n\}$ is bounded;
(b) Its weak limit x belongs to the closed linear subspace spanned by $\{x_1, x_2, \dots, x_n, \dots\}$;
(c) $\|x\| \leq \lim_n \|x_n\|$

Proof: The assertion (a) follows from theorem (2.1.3) next if (b is not true, then by theorem (1.3) there is an $x^* \in X^*$ such $x^*(x_n) = 0$ or all $n \geq 1$, but $x^*(x) \neq 0$. This is impossible, since $\{X_n\}$ converges weakly to x. finally, from $x^*(x) = \lim_n x^*(x_n)$ we get

$$|x^*(x)| = \lim_n |x^*(x_n)| \leq \lim_n\{\|x^*\|\|x_n\|\} = \|x^*\| \lim_n \|x_n\|$$

The next two results are concerned with important properties of reflexive spaces.

Theorem 2.1.8 [5]: Let X be a reflexive Banach space. A set $K \subset X$ is weakly sequentially compact if and only if it is both bounded and weakly closed.

Proof: Suppose K is weakly sequentially compact. Then K is weakly closed. Indeed, let $\{x_n\}$ be weakly convergent to x, there is a subsequence $\{X_{n_1}\}$ that is weakly convergent. But then, by theorem (2.1.7)(a), $\{X_{n_1}\}$ is bounded, which is impossible since $\|X_{n_1}\| > n_1$.

Suppose now that K is bounded and weakly closed. Let $\{X_n\}$ be a (bounded) sequence in K. We shall extract a weakly convergent subsequence. Denote by Z the closed linear subspace spanned by $\{x_1, x_2, \dots, x_n, \dots\}$. Z is clearly separable and by theorem (2.1.5), it is reflexive. Hence, Z^{**} is separable. Theorem (2.1.1) implies that Z^* is separable let $\{x_n^*\}$ be a dense sequence in Z^*. Since the sequence $\{x^*(x_n)\}$ is bounded, we can extract a convergent subsequence $\{x^*(x_{n,1})\}$. Next we extract from the sequence $\{x(x_{n,1})\}$ a convergent subsequence $\{x^*(x_{n,2})\}$. We proceed in this way step by step. In the kth step we extract a convergent subsequence $\{x_k^*(y_k)\}$ exists for any x_n^*. Since $\{x_n^*\}$ is dense in Z^* and since $\{y_k\}$ is a bounded sequence in Z it follows by that $\lim_k z^*(y_k)$exists for each $z^* \in Z^*$. Hence, $\lim_k (ky_k)\,(z^*)$ exists for each $z^* \in Z^*$. It follows that there exists a $y^{**} \in Z^{**}$ such that $\lim_k (ky_k)(z^*) = y^{**}(z^*)$ for all $z^* \in Z^*$. Since Z is reflexive there is then a point $y \in Z$ such that its image in Z^{**} (by the natural imbedding) is y^{**}. Therefore,

$$\lim_k z^*(y_k) = z^*(y) \text{for any } z^* \in Z^*$$

Now take any $x^* \in X^*$, it determines an element $z^* \in Z^*$ by $z^*(z) = x^*(z)$ for all $z \in Z$. Since $\{y_k\}$ we conclude that

$$\lim_k x^*(y_k) = x^*(y).$$

Hence $\{y_k\}$ is weakly convergent to y. Since K is weakly closed $y \in K$. Thus K is weakly sequentially compact

Theorem 2.1.9 [13]: A reflexive normed linear space X is weakly complete

Proof: Let $s\{x_n\}$ be a weak Cauchy sequence in X. Then $\lim_n x^*(x_n)$ exists for all $x^* \in X$. By theorem (2.1.7), $\{x_n\}$ which is bounded, from the proof of theorem (2.1.8) it then follows that there is a subsequence $\{X_{n_1}\}$ is bounded that converges weakly to some element $x \in X$. But then

$$\lim_n x^*(x_n) = \lim_{nj} x^*(x_{n1}) = x^*(x) \text{for any } x^* \in X$$

Thus, $\{X_n\}$ is weakly convergent to x.

Theorem 2.1.10 [14]: Let X be a normed linear space and let $\{x_n\} \subset X$. If $\{x_n\}$ converges weakly to y, then there exists a sequence $\{\sum \lambda_{1-n} x_1\}_{j=1}^{m}$ (with λ_{j-n} scalars) that converges to y.

Theorem 2.1.11 [15]: Let T be a bounded linear operator from a normed linear space X into a normed linear space Y. If $\{x_n\}$ is a sequence in X that is weakly convergent to x_0, then $\{Tx_n\}$ (in Y) is weakly convergent to Tx_0.

Theorem 2.1.12 [16]: A sequence $\{x_n\}$ in a normed linear space X is weakly convergent to x_0 if and only if the following conditions hold: (i) the sequence $\{2x_{n2}\}$ is bounded and (ii) $x^*(x_n) \to x^*(x_0)$ as $n \to \infty$, for any x^* in a set B dense in X^*.

Theorem 2.1.13 [17]: Let K be a weakly closed set in a reflexive Banach space X. Then there exists an element $x \in K$ such that $\inf_{x \in K} \|x\| = \|x\|$.

2.2 Tychonoff's theorem

Let X be a topological space and let χ be the class of its open sets. A subclass χ_0 of χ is called a neighborhood basis for the topology if every set in χ is a union of sets of χ_0. Recall that a neighborhood of a point x is a subclass $\chi_0(x)$ of χ having the property that every neighborhood of x contains a set of $\chi_0(x)$.

Suppose we are given every $x \in X$ a neighborhood basis $\chi_0(x)$at x. Then the following properties are clearly satisfied:

(1) $x \in W(x)$ for every $W(x) \in \chi_0(x)$
(2) The intersection $W_1(x) \cap W_2(x)$ of two sets $W_1(x), W_2(x)$ of $\chi_0(x)$ contains a set of $\chi_0(x)$.
(3) If $y \in W(x), W(x) \in \chi_0(x)$, then there is a set $W(y)$ of $\chi_0(y)$ such that $W(y) \subset W(x)$.

Conversely, if for any $x \in X$ there is assigned a class $\chi_0(x)$ of sets of X such that (1)-(3) hold, then a topology can be defined on X by taking the open sets to be all the possible union of sets $W(x)$ in $\chi_0(x)$ with $x \in X.T$. Note that each set $\chi_0(x)$ is then a neighborhood basis at x.

Lemma 2.2.1 [18]: A topological space X is compact if and only if every class $\{F_\alpha\}$ of closed sets in X with empty intersection contains a finite subclass $\{F_{1j}, \dots, F_{\alpha j}\}$ with empty intersection.

The proof is obtained by noting that the sets $F_\alpha^c = X - F_\alpha$ are open and using the relations:

$(U_\alpha F_\alpha^c) = \cup_{j=1}^{n} F_{\alpha j}\left(U_{\alpha j} F_{\alpha j}^c\right) = \cup_{j=1}^{n} F_{\alpha j}$

Lemma 2.2.2 [3]: A closed subset of a compact topological space is compact.

Proof: Let B be a closed subset of a compact subset of a compact topological space X. Since a closed subset of a B is a closed subset of X, Lemma 3.1 yields the assertion. A class $\{F_\alpha\}$ of sets of X is called centralized (or said to have the finite intersection properly) if any finite number of sets F have nonempty intersection. Lemma 3.1 immediately yields:

Lemma 2.2.3 [7]: A topological space X is compact if and only if every centralized class of closed sets has a nonempty intersection.

A class $\{M_\alpha\}$ of sets is said to have a common point of contract $U_\alpha \overline{M}_\alpha \neq \emptyset$. Any point x that belongs to $U_\alpha \overline{M}_\alpha$ is called a point of common contract. We gives a slightly different version of lemma 3.3, the proof of which is left to the reader.

Lemma 2.2.4 [20]: A topological space X is compact if and only if every centralized class of sets in X has a common point of contract.

A sequence $\{X_n\}$ in a topological space X is said to convergent to x if for any neighborhood U of x, there is an n_0 such that $x_n \in U$ for all $n \geq n_0$. If every sequence in X has a subsequence that is convergent to some point of X. Then we say that X is sequentially compact, X is said to satisfy the first countability axiom if there exists a countable neighborhood basis at each point of x.

Theorem 2.2.5 [11]: Let X be a topological space satisfying the first countability axiom. If X is compact, then it is also sequentially compact.

Proof: Let $\{X_n\}$ be a sequence in X. For every positive integer n, let $M_n = \{X_n, X_{n+1}, \dots,\}$. The class $\{M_n\}$ is centralized. Hence, by lemma 2.2.4, there is a point $\bar{x}$ of common contact that is $\bar{x} \in U_{n=1}^{\infty} \overline{M}_n$. Take a countable basis of neighborhoods $U_m$at $\bar{x}$. the subsequence $\{X_{n,1}\}$ of $\{X_n\}$ consisting of all points that belong to U_1 must be infinite. Indeed, otherwise $U_1 \cap M_n = \emptyset$ for some n sufficiently large. But this is impossible, since $\bar{x} \in \overline{M}_n$. By the same argument, the subsequence $\{x_{n,2}\}$of $\{x_n\}$, consisting of all points that belong to $U_1 \cap U_2$ is infinite. Note that $\{x_{n,2}\}$ is a subsequence of $\{X_{n,1}\}$. We proceed in this way of step 8. In the kth step we get an infinite subsequence $\{x_{n,k}\}$ that is contained in $U_1 \cap U_2 \cap \dots \cap U_k$. It is now easily seen that sequence $\{x_{n,k}\}$ converges to $\bar{x}$. let$X_\alpha(\alpha \in A$ an ordered set) be topological spaces, let

$$X = \prod_{\alpha \in A} X_\alpha$$

be the space whose element are the ordered sets $x = \{X_\alpha\}$. We define on X a topology by giving a neighborhood basis $\chi_0(x^0)$ at each point $x^0 = \{X_\alpha^0\}$. A set $U(x^0)$ is in $\chi_0(x^0)$ if there exists a finite number of indices $\alpha_1, \dots, \alpha_n$ from A and neighborhood $U(x_{\alpha_1}^0)$ of $x_{\alpha_1}^0$ in X_{α_1} such that $U(x^0)$ is the set of all points $\{x_\alpha\}$ with $x_{\alpha_1} \in U(x_{\alpha_1}^0)$ for $I = 1, \dots, n$ and x_α arbitrary in X_α if $\alpha \neq \alpha_1, \dots, \alpha \neq \alpha_n$. It is easy to verify that the conditions (1)-(3) for neighborhood bases are satisfied. Thus, X is a topological space with the classes $\chi_0(x^0)$ as neighborhood bases. We call X the Cartesian product of the topological spaces X_α. Consider the function $f_\alpha(x) = x_\alpha$ from Xonto X_α. The image of a set $M \subset S(X)$ under this map is called the projection of Mon X_α. The next result is called the Tychonoff theorem.

Theorem 2.2.6 [8]: The Cartesian product of any family of compact topological spaces is a compact topological space.

Proof: Let $X = \prod X_\alpha$ be the Cartesian product of the compact topological spaces X_α. In view of lemma 2.2.3 it suffices to show that every centralized class of sets $\{K^y\}$ has at least one point of common contact. Consider all the centralized classes of sets $\{N^\mu\}$ that contain $\{K^y\}$ and define a partial ordering: $\{N^\mu\} \leq \{M^\lambda\}$ if each set N^μ coincides with some set M^λ. We can apply Zorn's lemma and thus conclude that there exists a maximal centralized class $\{M^\lambda\}$containing $\{K^y\}$. It suffices to show $\{M^y\}$ has at least one point of common contact. The maximality of $\{M^\lambda\}$ implies:

(a) The intersection of any finite number of set of $\{M^\lambda\}$ is again in $\{M^\lambda\}$.
(b) If a set intersects any finite number of set of $\{M^\lambda\}$, then it belongs to $\{M^\lambda\}$.
(c) Indeed, if B is an intersection of a finite number of set of $\{M^\lambda\}$, then the class $\{M^\lambda B\}$ is centralized. By the maximality of $\{M^\lambda\}$ we conclude that $\{M^\lambda, B\} = \{M^\lambda\}$. That is, $B \in \{M^\lambda\}$. The proof of (b) is similar.

Denote the projection of a set M^λ on $X_\alpha$by M_α^λ. For any $\alpha, \{M_\alpha^\lambda\}$ is a centralized class in X_α. Since X_α is compact the sets $\{M_\alpha^\lambda\}$ have a point x_α^0 of common contact. Let $x^0 = \{x_\alpha^0\}$, we shall prove that x^0 is a point of common contact of class $\{M^\lambda\}$- that is, every neighborhood $U(x^0)$ of x^0 in a neighborhood basis at x^0 intersects every set M^λ.

By definition $U(x^0)$ consists of points $\{x_\alpha\}$, where $x_{\alpha j} \in U(x_{\alpha j}^0)$ for some indices $\alpha_1, \dots, \alpha_n$ and x_α varies in X_α for all $\alpha \neq \alpha_1, \dots, \alpha_n$. Here $U(x_{\alpha j}^0)$ is a neighborhood of $x_{\alpha j}^0$in $X_{\alpha j}$. Denote by $U(x^0)$ the neighborhood of x^0 defined by $x_{\alpha j} \in U(x_{\alpha j}^0)$ and x_α varies in X_α for all $\alpha \neq \alpha_1$. Then,

$$U(x_0) = U_{i=1}^n U_i(x_0)$$

Since, $x_{\alpha j}^0$ is a point of contact of the class $\{M_{\alpha j}^\lambda\}, U(x_{\alpha j}^0)$ intersects each $M_{\alpha j}^\lambda$. Hence, $U_i(x^0)$ intersects each set $\{M^\lambda\}$. But then (b) implies that $U_i(x^0)$ is in $\{M^\lambda\}$. Hence, by (a), $U(x^0)$ is in $\{M^\lambda\}$. It follows that $U(x^0)$ interest each set M^λ.

2.3 Weak Topology in Conjugate Spaces

Definition 2.3.1 [8]: A set X is called a topological linear space if:

(i) X is a linear vector space
(ii) X is a Hausdorff topological space
(iii) The function $(x, y) \to x + y$ from $X \times X$ into X is continuous
(iv) The function $(\lambda, x) \to \lambda x$ from $F \times X$ into X is continuous

Here F is either Ror C.

Note that metric linear spaces and in particular, normed linear spaces are topological linear spaces. Let X be a normed linear space and let X^* be its conjugate. We shall introduce a new topology on X^*, different from the Banach-space topology will be given by neighborhood bases $\chi_0(y^*),\ y^* \in X^*$. Then sets of $\chi_0(y^*)$ are given by

$$N(y^*; x_1, \dots, x_n;\ \varepsilon) = \{x^*; |x^*(x_j) - y^*(x_j)| < \varepsilon\ for\ 1 \leq I \leq n\}$$

Where ε is any positive number, n is any positive integer and $x_1, \dots, x_n$ are any points of X.

Theorem 2.3.1 [14]: The neighborhood bases $\chi_0(y^*)$ satisfy the conditions (1)-(3) of section

4.11 and, therefore determine a topology on $X^*$with this topology, X^* is a topological space.

Definition 2.3.2 [19]: The topology introduced by the neighborhood bases $\chi_0(y^*)$ is called the weak topology.

Proof: The condition (1) is obvious, the condition (2) follows by noting that the set

$\{x^*; |x^*(x_j) - y^*(x_j)| < \varepsilon'$ for $1 \le i \le n\} \cap \{x^*; |x^*(x_j) - y^*(x_j)| < \varepsilon$ for $1 \le I \le m\}$

Contains the set

$\{x^*; |x^*(z_i) - y^*(z_i)| < \varepsilon''$ for $1 \le i \le n + m\}$

Where $\varepsilon'' = \min(\varepsilon, \varepsilon'), z_i = x_i$ if $1 \le i \le n, z_{n+i} = x_i$if $1 \le i \le m$. To prove (3) let.

$y_1^* \in N(y^*; x_1, \dots, x_n; \varepsilon)$Then

$|y_1^*(x_i) - y^*(x_i)| < \varepsilon'if1 \le i \le n$

For some $0 < \varepsilon'\varepsilon$. It follows that

$N(y_1^*; x_1, \dots, x_n; \varepsilon - \varepsilon') \subset N(y^*; x_1, \dots, x_n; \varepsilon)$

Having proved (1)-(3), we conclude that X^* is a topological space having the sets $\chi_0(y^*)$ for neighbor

Bases. To prove that X^* is a Hausdorff space, let $y_1^* \neq y_2^*$. Then there exists an $x_0 \in X$ such that $y^*(x_0) \neq y^*(x_0)$. Hence, if $\varepsilon = |y_1^*(x_0) - y_2^*(x_0)| < \frac{1}{2}$, then the two neighborhoods

$N(y_1^*; x_0; \varepsilon), \qquad N(y_2^*; x_0; \varepsilon)$

are disjoint. We have thus proved the condition for topological space to be Hausdorff space. To complete the proof of theorem 2.3.1 we have to establish the properties (iii), (iv) in definition 2.3.2 To prove (iii), take any neighborhood $N = N(y^* + y^*; x_{i-1}, \dots, x_n; \varepsilon)$of $y^* + y^*$. It suffices to find neighborhoods $N_1 + N_2$ is contained in N. If we take

$$N_j = \left\{y_1^*; x_1, \dots, x_n; \frac{1}{2}\right\} (j = 1,2)$$

then, obviously $N_1 + N_2 \subset n$

To prove (iv) take any neighborhood $N' = N(\lambda_0 y^*; x_1, \dots, x_n; \varepsilon)$ of $\lambda_0 y^*$. It suffices to find a neighborhood M of λ_0 if F and a neighborhood N^n of y^* such that $\{\lambda x^*; \lambda \in M, x^* \in N^n\}$ is contained in N' let

$M = \{\lambda; |\lambda - \lambda_0| < \varepsilon'\}, \; N'' = N(y^*; x_1, \dots, x_n; \varepsilon')$
Then, if $\lambda \in N''$

$|\lambda x^*(x_i) - \lambda_0 y^*(x_i)| = |\lambda x^*(x_i) - \lambda y^*(x_i)| + |\lambda - \lambda_0||y^*(x_i)|$

$\le |\lambda|\varepsilon' + K\varepsilon' \le (|\lambda_0| + \varepsilon' + K\varepsilon')$

Where $K = \max|y^*(x_i)|$. Taking $\varepsilon' < 1, (|\lambda_0| + 1 + K)\varepsilon' < \varepsilon$, we find that $|\lambda x^*(x_i) - \lambda_0 y^*(x_i)| < \varepsilon$ for $1 \le I \le n$. Hence $\lambda x^* \in N'$. This completes the proof of (iv) and of the theorem.

Whereas in the norm topology bounded sets in X^* are not relatively compact in general, they are relatively compact in the weak topology. This will be proved in the next theorem, called the theorem of Alaogu. This result is one of the most important illustrations of the importance of the concept of weak topology.

Theorem 2.3.2 [15]: Let X be a normed linear space, then the closed unit balls in X^* is compact in the weak topology.

Proof: To each $x \in X$ we correspond the closed real interval $I_x = [-2x_2, 2x_2]$ if X is real space and the closed disc in the complex plane $I_x = \{z; |z| \le 2x_2\}$ if X is a complex space. Denote by I the Cartesian product $\prod_{x \in X} I_x$. By Tychonoff's theorem, I is compact.

Consider now the set $\Gamma = \{\mathfrak{f}\}$ of all elements $\mathfrak{f}$ of X^* with $2\mathfrak{f}2 \le 1$. We have to prove that Γ is compact. For each $\mathfrak{f} \in \Gamma, |\mathfrak{f}(x)| \le 2x_2$. Thus we can correspond to a point $\mathfrak{f}$ in I , having coordinates $\mathfrak{f}(x)$. We shall denote this correspondence by σ. Thus, σ$\mathfrak{f}$ = $\mathfrak{f}$. It is clear that σ is one to one. Let $I' = \sigma(\Gamma)$. From the definition of neighborhoods in X^* and I it follows that σ and its inverse are both continuous. Hence Γ is compact if and only if I' is compact.

To prove that $I'$is compact it suffices (by lemma 2.2.2) to show that $I'$is closed. Let $\mathfrak{f}_0$ be a point in I', we first show that the functional $\mathfrak{f}_0$ defined by $\mathfrak{f}_0(x) = \mathfrak{f}_0(x)$ (the $x -$ coordinate of $\mathfrak{f}_0$) is linear. Take any points x_1, x_2 in X and any scalars λ_1, λ_2. Let $U_0 = U(\mathfrak{f}_0; x_1 x_2, \lambda_1 x_1 + \lambda_2 x_2; \varepsilon)$ be a neighborhood of $\mathfrak{f}$ consisting of the points $\{y_x\}$ with

$$|yx_1 - \mathfrak{f}_0(x_1)| < \varepsilon, \; |yx_2 - \mathfrak{f}_0(x_2)| < \varepsilon, \; |y\lambda_1 x_1 + \lambda_2 x_2 - \mathfrak{f}_0(\lambda_1 x_1 + \lambda_2 x_2)| < \varepsilon$$

and arbitrary if $x \neq x_1,\ x \neq x_2,\ x \neq \lambda_1 x_1 + \lambda_2 x_2$. Since $\mathfrak{f}_0 \in I^{'}$, there is an element $\mathfrak{f}$ in $I^{'} \cap U_0$. It has the form $\mathfrak{f} = \sigma \mathfrak{f}$, where $\mathfrak{f}$ is a continuous linear functional. Thus,

$$|\mathfrak{f}(x_1) - \mathfrak{f}_0(x_1)| < \varepsilon, \quad |\mathfrak{f}(x_2) - \mathfrak{f}(x_2)| < \varepsilon,$$
$$|\mathfrak{f}(\lambda_1 x_1 + \lambda_2 x_2) - \mathfrak{f}_0(\lambda_1 x_1 + \lambda_2 x_2)| < \varepsilon$$

and

$$\mathfrak{f}(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \mathfrak{f}(x_1) + \lambda_2 \mathfrak{f}(x_2)$$

Consequently,

$$|\lambda_1 \mathfrak{f}_0(x_1) + \lambda_2 \mathfrak{f}_0(x_2) - \mathfrak{f}_0(\lambda_1 x_1 + \lambda_2 x_2)|$$
$$\leq |\lambda_1||\mathfrak{f}_0(x_1) - \mathfrak{f}(x_1)| + |\lambda_2||\mathfrak{f}_0(x_2) - \mathfrak{f}(x_2)| + |\mathfrak{f}_0(\lambda_1 x_1 + \lambda_2 x_2)|$$
$$- \mathfrak{f}(\lambda_1 x_1 + \lambda_2 x_2)| \leq (|\lambda_1| + |\lambda_2|)\varepsilon$$

Since ε is arbitrary, we get $\mathfrak{f}_0(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 \mathfrak{f}_0(x_1) + \lambda_2 \mathfrak{f}_0(x_2)$. Thus, the functional $\mathfrak{f}_0$ defined by $\mathfrak{f}_0(x) = \mathfrak{f}_0(x)$ is linear. Since $\mathfrak{f}_0(x) \in I_x, |\mathfrak{f}_0(x)| \leq 2x_2$. It follows that $\mathfrak{f}_0 \in X^*$and $2\mathfrak{f}_0 \leq 1$. Hence $\mathfrak{f}_0 \in \Gamma$, but then $\mathfrak{f}_0 = \sigma \mathfrak{f}_0 \in I'$. We have thus proved that $I^{'} = I^{'}$

Definition 2.3.2 [16]: Let X be a normed linear space. A sequence $\{X^*\}$ in X^* is said to be weakly convergent if there exists an element $x^* \in X^*$ such that $\lim x^*(x_n) = x^*(x)$ for all $x \in X$. We then call x^* the weak limit of $\{x^*\}$ and we say that $\{x^*\}$ is weakly convergent to x^*.

Note that a sequence $\{x_n^*\}$ cannot have two distinct weak limits

Theorem 2.3.3 [12]: Let X be a separable normed linear space, then every bounded sequence of continuous linear functional in X^* has a weakly convergent subsequence.

Proof: Denote by $X_R^*(R > 0)$ the topological subspace $\{x^* \in X^*; 2x_2 \leq R\}$ (with the weak topology). Its topology can be defined by the neighborhood bases consisting of the sets $X_R^* \cap N$, where N varies over the sets $N(x^*; x_1, \dots, x_n; \varepsilon)$.

Let $\{y_n\}$ be a dense sequence in X for any $z^* \in X_R^*$, the sets

$$X_R^* \cap N\left(z^*; y_{k1}, \dots, y_{kn}; \frac{1}{m}\right) (n = 1,2, \dots; m = 1,2, \dots)$$

from a countable neighborhood basis (for weak topology of X_R^*) at z^*. Thus, the topological linear space X_R^* satisfies the first countability axiom. Now let $\{x^*\}$ be a sequence of continuous linear functional with $2x_2^* \leq K,\ K$ constant.

By the Alaogu theorem, the set X_k^* is compact in the weak topology. By theorem 2.2.3, this set is then sequentially compact in the weak topology. It follows that there is a subsequence $\{x_n^*\}$ that is weakly convergent.

3. ADJOINT OPERATORS

Definition 3.1.1 [17]: Let X and Y be normed linear spaces and let $T \in R(X, Y)$. The adjointT^* of T is an operator from Y^* into X^* defined by $(T^* y^*)(x) = y^*(Tx)$.

Theorem 3.1.1: The mapping $T \to T^*$ is an isometric isomorphism of $R(X, Y)$into $R(Y^*, X^*)$.

Proof: Take any $y^* \in Y^*. T^* y^*$ is clearly a linear functional on X. Furthermore

$$|(T^* y^*)(x)| \leq \|y^*\|\|Tx\| \leq \|y^*\|\|T\|\|x\|$$

Hence,

$$\|T^*\| = \sup_{\|y^*\|=1} \|T^* y^*\| = \sup_{\|y^*\|=1} \sup_{\|y^*\|=1} |(T^* y^*)(x)|$$
$$= \sup_{2y^*2=1} |y^*(Tx)| = \sup_{2y^*2=1} |y^*(Tx)| = \sup_{\|y^*\|=1} \|Tx\|$$
$$= \|T\|$$

Theorem 3.1.2 [2]: Let X, Y, Z be normed linear spaces and let $T \in R(X, Y), S \in R(Y, Z)$. Then $(ST)^* = T^* S^*$. The adjoint of the identity 1 in $R(X)$ is the identity in $R(X^*)$.

Proof: $z^* \in Z^*$. Then, for any $x \in X$,

$$[(ST)^* z^*](x) = z^*[(ST)x] = z^*[S(Tx)]$$

$$= (S^* z^*)(Tx) = [T^*(S^* z^*)](x) = [(T^* S^*) z^*](x)$$

Thus $(ST)^* z^* = (T^* S^*) z^*$ for any $z^* \in Z^*$ – that is $(ST)^* = T^* S^*$. Next $(I^* x^*)(x) = x^*(Ix) = x^*(x)$ –that is, $I^* X^* = X^*$. Thus I^* is the identity in $R(X^*)$.

Denote by $\hat{X}, \hat{Y}$ the images under the natural imbedding of $\hat{X}, Y$ into X^{**}, Y^{**} respectively. If $T \in R(X, Y)$, then define $\hat{T} \in R(\hat{X}, \hat{Y})$ by $\hat{T}\hat{X} = \hat{y}$where $y = Tx$. If S is a linear operators from a set Ds in $\hat{X}^{**}$ into Y^{**} such that $Ds \supset \hat{X}$ and $S\hat{x} = Ds = \hat{X}$, then we write $S = T$.

Theorem 3.1.3 [8]: Let X, Y be normed linear spaces and let $T \in R(X, Y)$. Then the second adjoint$T^{**}; X^{**} \to Y^{***}$ is an extension of T. If X is reflexive, then $T^{**} = T$

Proof: Let $x \in X, y^* \in Y^*$, then

$$(T^{**}\hat{x})(y^*) = \hat{x}(T^* y^*) = (T^* y^*)(x) = y^*(Tx)$$

$= Tx(y^*) = (\hat{T}\hat{x})(y^*)$

That is, $T^{**}\hat{x} = \hat{T}\hat{x}$. this completes the proof

Theorem 3.1.4 [9]: Let X be a Banach space and let Y be a normed linear space. A linear operator $T \in R(X, Y)$ has bounded inverse T^{-1} (with domain Y) if and only if T^* has a bounded inverse $(T^*)^{-1}$ (with domain X^*). In that case, $(T^{-1})^* = (T^*)^{-1}$.

Proof: Suppose T^{-1} exists. By theorem 3.1.3, $I^* = (TT^{-1})^* = (T^{-1})^* T^*$, where I^* is the identity operator in $R(Y^*, Y^*)$. Similarly, $T^*(T^{-1})^*$ is the identity operator in $R(X^*, X^*)$. It follows that $(T^*)^{-1}$ exists and is equal to $(T^{-1})^*$.

Suppose conversely that $(T^*)^{-1}$ is a bounded operator in $R(X^*, Y^*)$. Then by what we have just proved $(T^{**})^{-1}$ exists and s is in $R(Y^{**}, X^{**})$. By theorem 3.1.3, T^{**} is an extension of T. Hence T is one to one. If we can prove that $T(X) \neq Y$. Then $T(Y) \neq X$, there exists a $y^* \in Y^*$ with $y^* \neq 0$, $y^*(Tx) = 0$ for all $x \in X$. Hence, $T^* y^* = 0$, this is impossible since T^* is one to one

Definition 3.1.2 [2] Let X be a normed linear space and let A be a subset of X. We shall denote by $A^{'}$ the set of all Γ ⊂ X* we define the orthogonal complement Γ of Γ to be the set of all $x \in X$ such that $x^*(x) = 0$ for all $x^* \in \Gamma$. Notice that A_1 and Γ_1 are closed linear subspaces in $X^*$and X, respectively. Let $T \in R(X, Y)$. We shall denote by N_T the null space of T – that is, $N_T = \{x; Tx = 0\}$. Similarly, we denote by N_T, the space of T^*. Finally, we denote by R_T the range of an operator T. Thus if $T \in R(X, Y)$, then $R_T = T(x)$.

Theorem 3.1.5 [16]: Let X, Y be normed linear spaces and let $T \in R(X, Y)$. Then

$$\bar{R}_T = N_T^1 \qquad (3.1.1)$$

Proof: We first prove that $\bar{R}_T \supset N_T^1$, what we have to show is that if $y_0 \notin N_T^1$ there exists a $y^* \in Y^*$ such that $y^*(y_0) \neq 0$ and $y^*(Tx) = 0$ for all $x \in X$. Hence,$T^* y^* = 0$. Since

$N_2^1 = \{y \in Y; y^*(y) = 0$ for y^* such that $T^* y^* = 0\}$,

It follows that $y_0 \in N_T^1$.

We next prove that $\bar{R}_T \subset N_T^1$. Let $y \in \bar{R}_T$. Then there exists a sequence $\{y_n\} \in R_T$ such that $\lim y_n = y$. If $T^* y^* = 0$, then $y^*(y_n) = y^*(Tx_n) = (T^* y^*)(x_n) = 0$, where $Tx_n = y_n$. Hence also $y^*(y) = 0$. We conclude that $y \in N_T^1$

Remark: Theorem 3.1.3 can be interpreted as an existence theorem for the equation $Tx = y$. It then implies the following result: if T^* is one to one and R_T is closed, then $R_T = Y$ – that is for any given $y \in Y$ there exists a solution x in X of the equation $Tx = y$. The dual of (3.1.1) is $\bar{R}_T = N_T^1$. But this relation is not true in general. We shall prove, however the following weaker form of it.

Theorem 3.1.6 [5]: Let X and Y be Banach spaces and let $T \in R(X, Y)$.

$$R_T = N_T^1 \qquad (3.1.2)$$

Lemma 3.1.7 [19]: Let X, Y be Banach spaces and let $T \in R(X, Y)$. If R_T is closed, then there exists a constant K such that for any $y \in R_T$ there is a point $x \in X$, $|g(y)| = \|x^*\| Tx = y$and $\|x\| \leq K\|y\|$.

Proof: By the open mapping theorem T maps the unit ball B_1 of X onto a set containing a ball in R_T with center 0, that is
$TB_1 \supset \{y; y \in R_T, 2y_2 < \delta\}$for some $\delta > 0$

Now let $0 \neq y \in R_T$, then $\delta y / 2\|y\|$ is in TB_1 – that is $Tz = \delta y / 2\|y\|$ for some z, $\|z\| < 1$. Let $x = 2\|y\| z / \delta$. Then $Tx = y$and $\|x\| < (2/\delta)\|y\|$.

Proof of Theorem 3.1.6: We first prove that $N_T^1 \subset R_T^*$. Let $x^* \in N_T^1$, define a linear functional g on R_T by $g(Tx) = x^*(x)$. g is defined unambiguously. Indeed, if $Tx_1 = Tx_2$, then $(x_1 - x_2) \in NT$ and therefore $x^*(x_1 - x_2) = 0$. Hence, $g(Tx_1) - g(Tx_2) = x^*(x_1) = x^*(x_2) = 0$. By Lemma 3.1.7 there exists a constant

K such that for any $y \in R_T$ there is an x with $Tx = y, \|x\| \leq K\|y\|$. Hence,

$|g(y)| = |x^*(x)| \leq K\|x^*\|\|y\|$

Thus g is a bounded linear functional on R_T. By Hahn-Banach theorem we can extend g into a continuous linear $y^* \in Y^*$. Since for any $x \in X$

$T^* y(x) = y^*(Tx) = g(Tx) = x^*(x^*)$

We have $T^* y^* = x^*$, thus $x^* \in R_{T^*}$

Suppose conversely that $x^* R_{T^*}$, we shall prove that $x^* \in N_T$. Let $y^* \in Y^*$ be such that $T^* y^* = x^*$. if$x \in N_T$, then

$x^*(x) = (T^*y^*) = y^*(Tx) = y^*(0) = 0$

This shows that $x^* \in N_T^1$

The assertion that R_T is closed follows (3.1.2) and the fact that N is closed.

3.1 The Conjugates of L^∞ and $C[0,1]$

Let (X, μ) be a measure space. we shall denote by $\|\mathfrak{f}\|$, the norm of $\mathfrak{f}$ in the space $L^p(X,\mu)$that is

$$\|\mathfrak{f}\|_p = \left\{\int |\mathfrak{f}|^p \, d\mu\right\}^{1/p} \qquad (1 \le p < \infty)$$

We also write

$$\|\mathfrak{f}\|_\infty = ess \sup_x |\mathfrak{f}|$$

Theorem 3.2.1:[6]Assume that (X, μ) is a $\sigma-$finite measure space and let $1 < 1/p < \infty$. Then to every continuous linear functional x^* on $L^p(X,\mu)$ there corresponds a unique element g in $L^q(X,\mu)$ such that

$$x^*(\mathfrak{f}) = \int \mathfrak{f}g \ d\mu \text{for all } \mathfrak{f} \in L^p(X,\mu) \qquad (3.2.1)$$

Furthermore,

$$\|x^*\| = \|g\|_q \qquad (3.2.2)$$

Proof: we shall give the proof only for the complex $L^p(X,\mu)$ space. the proof for the real space is similar. The uniqueness of g is obvious. We shall need the following lemma:

Lemma 3.2.2 [8]: Let $\mathfrak{f}$ be a measurable function

(a) If $\mathfrak{f} \in L^p(X,\mu)$, then
(b) $\|\mathfrak{f}\|_p = \max_{\|g\|_p=1} | \int \mathfrak{f}g \, d\mu | = max_{\|g\|_p=1} \int |\mathfrak{f}g| \, d\mu$ (3.2.3)
(c) If $\mathfrak{f} \notin L^p(X,\mu)$, then $\sup_{\|g\|_p=1} \int |\mathfrak{f}g| \, d\mu = \infty$

Proof: To prove (a) we may clearly assume that $\|\mathfrak{f}\|_p \neq 0$. By Holder's inequality,

$$\int |\mathfrak{f}g| \, d\mu \le \|\mathfrak{f}\|_p = \|\mathfrak{f}\|_p \text{if} \|g\|_q = 1 \qquad (3.2.4)$$

Next let $g = \theta(\mathfrak{f})|\mathfrak{f}|\|\mathfrak{f}\|$ –where $\theta(\mathfrak{f})$ is defined by $\mathfrak{f} - (\mathfrak{f}), |\mathfrak{f}|$ if $\mathfrak{f} \neq 0$ and $\theta(\mathfrak{f}) = 0$if $\mathfrak{f} = 0$. It is easily seen that $\theta(\mathfrak{f})$ is measure. We have

$$\int |g|^q \, d\mu = \int |\mathfrak{f}|^{(p-1)q} \, \|\mathfrak{f}\|_q^{-p} d\mu = 1$$

Also

$$\int \mathfrak{f}g \ d\mu = \int |\mathfrak{f}|^p \, \|\mathfrak{f}\|^{-p/q} d\mu = \|\mathfrak{f}\|^{p-p/q} = \|\mathfrak{f}\|_p$$

This together with (6.4), completes the proof of (a).

To prove (b), write $X = X_n$, where $X_n \subset X_{n+1}, \mu(X_n) < \infty$, and introduced for any positive integer n, the function

$$\mathfrak{f}n(x) = \begin{cases} \mathfrak{f}(x), & if\ |\mathfrak{f}(x)| \le n, \quad x \in X_n \\ n, & if\ |\mathfrak{f}(x)| \le n, \ x \in X_n \\ 0, & if\ x \notin X_n \end{cases}$$

Then $x \in L^p(N,\mu)|\mathfrak{f}_n| \le |\mathfrak{f}|$ a.e. and $\|\mathfrak{f}_n\|_p \to \infty$ as $n \to \infty$. Hence,

$$\sup_{\|g\|_q=1} \int |\mathfrak{f}g| \, d\mu \ge \max_{\|g\|_q=1} \int |\mathfrak{f}_n g \,| \, d\mu = \|\mathfrak{f}_n\|_p \to \infty \text{as} n \to \infty$$

Corollary 3.2.1 [21]: Let $g \in L^p(X,\mu)$. Then the functional

$$G(\mathfrak{f}) = \int \mathfrak{f}g \ d\mu \quad [\mathfrak{f} \in L^p(X,\mu)]$$

Is a continuous linear functional on $L^p(X,\mu)$and $\|G\| = \|g\|_p$. It is obvious that G is linear, the equality $\|G\| = \|g\|_q$ follows from Lemma 6.2(a) with p and q interchanged. We proceed with the proof and we first make the assumption that $\mu(X) < \infty$. For any measurable set E, let

$$v(E) = x^*(x_E)$$

$v$obviously is finitely additive. We shall prove that it is completely additive. Let E_n be a sequence of mutually disjoint measurable sets and let $S_n = \cup_{k=1}^n E_1 S = \cup_{k=1}^n E_k$.

By the Lebesgue bounded convergence theorem [here we use the assumption that $\mu(X) < \infty$] we get

$$\|Xs_n - Xs\| \to 0 \text{as} n \to \infty$$

Hence, $x(Xs) = x^*(Xs)$, it follows that

$$v(S) = x^*(Xs) = \lim_n x^*(Xs_n) = \lim_n v(S_n)$$
$$= \lim_n \sum_{m=1}^\infty v(E_m) = \sum_{m=1}^\infty v(E_m)$$

We proved thus proved that v is a complex measure. If $\mu(E) = 0$, then X_E is zero as element in $L(X,\mu)$. Hence, $v(E) = x^*(X_E) = x^*(0) = 0$. Thus the complex measure v is absolutely continuous with respect to the measure μ . Finally, v is a finite complex measure, since any function XE is in $L(X\mu)$ and thus

$$|v(E)| = |X^*(X_E)| < \infty$$

By the Radon-Nikodym theorem there exist an integrable function g such that

$$v(E) = \int_E g d\mu$$

For any measurable set E. Hence

$$x(X_E) = \int_{X_E} g d\mu$$

By linearity of both sides with respect to X_E we get

$$x^*(\mathfrak{f}) = \int \mathfrak{f} g d\mu \qquad (3.2.5)$$

For any simple function $\mathfrak{f}$. We shall extend this relation to any bounded measurable function $\mathfrak{f}$. Writing $\mathfrak{f}$ in the form $\mathfrak{f}_1 - \mathfrak{f}_2 + i\mathfrak{f}_3 - i\mathfrak{f}_4$. Where the $\mathfrak{f}_i$ are bounded, nonnegative measurable functions, we conclude that there exists a sequence $\{\mathfrak{f}_n\}$ of uniformly Lebesgue bounded convergence theorem it follows that

$$\lim_n \int \mathfrak{f}_n g \, d\mu = \int \mathfrak{f} g \, d\mu$$

Since (by the Lebesgue bounded convergence theorem) $\|\mathfrak{f}_n - \mathfrak{f}\|_p \to 0$ as $n \to \infty$, we also have

$$\lim_n x^*(\mathfrak{f}_n) = x^*(\mathfrak{f})$$

Combining the last two relations with the fact that (6.5) holds with $\mathfrak{f}$ replaced by each $\mathfrak{f}_n$, we obtain the equality for the bounded measurable function $\mathfrak{f}$.

We next show that $g \in L^p(X,\mu)$ and define

$$\mathfrak{f}_n(x) = \begin{cases} |\mathfrak{f}(x)|\theta(g), & if\ |x| \le n \\ 0, & if |x| \le n \end{cases}$$

Then $\|\mathfrak{f}_n\|_p \le \|\mathfrak{f}\|_p$ and therefore

$$|x^*(\mathfrak{f}_n)| \le \|x^*\| \|\mathfrak{f}_n\|_p \le \|x^*\| \|\mathfrak{f}\|_p \qquad (3.2.6)$$

Since $\mathfrak{f}_n g \ge 0$ and $\lim \mathfrak{f} n g = |\mathfrak{f} g|\ a.e.$, Fatou's lemma gives

$$\int |\mathfrak{f} g| \, d\mu \le \lim \int \mathfrak{f}_n g d\mu = \lim x^*(\mathfrak{f} n) \le \|x^*\| \|\mathfrak{f}\|_p$$

COMPETING INTERESTS

Author has declared that no competing interests exist.

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